NANYANG TECHNOLOGICAL UNIVERSITY

ON RANKS OF PARTITIONS AND CONGRUENCES OF SPECIAL FUNCTIONS

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Division of Mathematical Sciences

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List of Works

Below is the list of work done during my PhD studies in NTU.

1. S. H. Chan, R. Mao, *Two congruences for Appell-Lerch sums*, Int. J. Number Theory **8** (2012), no. 1, 111–123.

2. S. H. Chan, R. Mao, *Pairs of partitions without repeated odd parts*, J. Math. Anal. Appl. **394** (2012), no. 1, 408–415.

3. R. Mao, *Some new asymptotic expansions of certain partial theta functions*, Ramanujan J., accepted.

4. R. Mao, *Ranks of partitions modulo* 10, J. Number Theory, **133** (2013), no. 11, 3678–3702.

5. R. Mao, *Inequalities between rank moments of overpartitions*, J. Number Theory, **133** (2013), no. 11, 3611–3619.

6. R. Mao, *M_2-ranks of partitions without repeated odd parts modulo* 6 *and* 10, submitted.

7. R. Mao, *Asymptotic inequalities for K-ranks and their cumulation functions*, J. Math. Anal. Appl., **409** (2014), no. 2, 729–741.

Abstract

This thesis focuses on the rank statistic of partition functions, congruences and relating identities of special functions such as Appell-Lerch sums and partition pairs. Most results in Chapter 2, 3, 4, 5 are reproduced from [58], [59], [25], [24], respectively.

F. J. Dyson conjectured that the rank of partitions provides combinatorial interpretations of S. Ramanujan's famous congruences for partition functions modulo 5 and 7. This together with some other identities between ranks of partitions modulo 5 and 7 were proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer. In Chapter 2, we prove identities for ranks of partition modulo 10. With a similar method, we obtain identities between the M_2-rank of partitions without repeated odd parts modulo 6 and 10 in Chapter 3.

A series of identities and congruences of Appell-Lerch sums were discovered by S. H. Chan recently. In Chapter 4, we give a generalization of Chan's results and also find a new series of identities for Appell-Lerch sums. As special cases, we prove congruences for some mock theta functions.

In Chapter 5, we prove two identities related to overpartition pairs. One of them gives a generalization of an identity due to J. Lovejoy, which was used in a joint work by K. Bringmann and Lovejoy to derive congruences for overpartition pairs. We apply our two identities of pairs of partitions where each partition has no repeated odd parts. We also present three partition statistics that give combinatorial explanations of a congruence modulo 3 satisfied by these partition pairs.

Contents

Chapter 1

Introduction

In this thesis, we study three related topics. Both Chapters 2 and 3 are on the same topic, rank differences of partition functions. In Chapter 4, we obtain certain identities and congruences of some Appell-Lerch sums. We prove identities related to overpartition pairs and study some combinatorial statistics of partition pairs without repeated odd parts in Chapter 5. Some open problems are given in Chapter 6. Most results of Chapter 2, 3, 4, 5 are reproduced from [58], [59], [25], [24], respectively.

A partition of a positive integer n is a sequence of non-increasing positive integers whose sum equals n and $p(n)$ is defined to be the number of partitions of n while $p(0) := 1$. The following three famous congruences for $p(n)$ were found and later proved by S. Ramanujan:

$$p(5n+4) \equiv 0 \pmod{5} \tag{1.0.1}$$

$$p(7n+5) \equiv 0 \pmod{7} \tag{1.0.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.0.3}$$

Indeed, Ramanujan [63] found the generating functions for $p(5n+4)$ and $p(7n+5)$,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.0.4}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{1.0.5}$$

Here and for the rest of the thesis, we use the notations,

$$\begin{aligned}
(x)_0 :=& (x;q)_0 := 1,\\
(x)_n :=& (x;q)_n := \prod_{k=0}^{n-1}\left(1-xq^k\right),\\
(x)_\infty :=& (x;q)_\infty := \prod_{k=0}^{\infty}\left(1-xq^k\right),\\
(x_1,\ldots,x_m)_\infty :=& (x_1,\ldots,x_m;q)_\infty := (x_1;q)_\infty \ldots (x_m;q)_\infty,\\
[x_1,\ldots,x_m]_\infty :=& [x_1,\ldots,x_m;q]_\infty := (x_1,q/x_1,\ldots,x_m,q/x_m;q)_\infty,\\
J_{a,b} :=& (q^a,q^{b-a},q^b;q^b)_\infty,\\
\overline{J}_{a,b} :=& (-q^a,-q^{b-a},q^b;q^b)_\infty,\\
J_b :=& (q^b;q^b)_\infty,\\
\overline{J}_b :=& (-q^b;q^b)_\infty,
\end{aligned}$$

and we assume $|q|<1$ for convergence.

In order to explain Ramanujan's congruences combinatorially, in 1944, F. J. Dyson [30] defined the *rank of a partition* to be the largest part minus the number of parts. If we denote $N(m,n)$ to be the number of partitions of n with rank m and $N(t,l,n)$ to be the number of partitions of n with rank congruent to t modulo l, then Dyson conjectured that

$$N(k,5,5n+4)=\frac{p(5n+4)}{5},\quad 0\le k\le 4 \tag{1.0.6}$$

and

$$N(k,7,7n+5)=\frac{p(7n+5)}{5},\quad 0\le k\le 6. \tag{1.0.7}$$

Since $p(5n+4)=\sum_{k=0}^{4}N(k,5,5n+4)$ and $p(7n+5)=\sum_{k=0}^{6}N(k,7,7n+5)$, we see that (1.0.6) and (1.0.7) imply (1.0.1) and (1.0.2), respectively. Thus Dyson's rank gives combinatorial interpretations for Ramanujan's first two congruences. Dyson's conjectures (1.0.6) and (1.0.7) were first proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [12]. They established the generating functions for every rank difference $N(b,l,ln+d)-N(c,l,ln+d)$ with $l=5$ or 7 and $0\le b,c,d<l$. Many of them were trivially 0, while others were infinite products and still others were generalized Lambert series. However, Dyson also observed that the corresponding analogue of (1.0.6) and (1.0.7) for $p(11n+6)\equiv 0 \pmod{11}$ does not hold. Hence he conjectured the existence of another partition statistic which he called the crank that would then give a combinatorial interpretation of Ramanujan's congruence (1.0.3). The crank of partitions was found by G. E. Andrews and F. G. Garvan in [32] and [6].

Motivated by the work of Atkin and Swinnerton-Dyer, we study the rank of partitions modulo 10 in Chapter 2. We consider the two groups of linear combinations of such ranks,

$$N(0,10,5n+d)+N(1,10,5n+d)-N(4,10,5n+d)-N(5,10,5n+d)$$

and

$$N(1,10,5n+d)+N(2,10,5n+d)-N(3,10,5n+d)-N(4,10,5n+d),$$

where $0 \leq d \leq 4$. It turns out that both groups of combinatorial objects given above have nice generating functions. Similar to the results in [12], many of the generating functions we found are either infinite products or sums of an infinite product and a generalized Lambert series. With these generating functions, inequalities between ranks modulo 10 are also discovered in Chapter 2.

Although Dyson's rank fails to explain Ramanujan's congruence (1.0.3) combinatorially, the method developed by Atkin and Swinnerton-Dyer [12] is widely used to obtain rank differences for other types of partition ranks (see [55–57], for example). In particular, the M_2-rank differences of partitions without repeated odd parts were obtained by J. Lovejoy and R. Osburn in [56]. Recall that the M_2-rank was first introduced by A. Berkovich and Garvan in [13]. They defined the M_2-rank of a partition λ without repeated odd parts as

$$M_2\text{-rank}(\lambda)=\left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda),$$

where $l(\lambda)$ is the largest part of λ, $n(\lambda)$ is the number of parts of λ and $\lceil . \rceil$ is the ceiling function. Let $N_2(t,l,n)$ denote the number of partitions of n without repeated odd parts whose M_2-rank is congruent to t modulo l and for $0 \leq b,c,d \leq l$, define

$$R_{bc}(d) := \sum_{n\geq 0} \left(N_2(b,l,ln+d) - N_2(c,l,ln+d)\right) q^n.$$

Then Lovejoy and Osburn gave all the generating functions for $R_{bc}(d)$ for $l=3$ and 5. Motivated by the work in [56] and with a similar method in Chapter 2, we study the M_2-rank of partitions without repeated odd parts modulo 6 and 10 in Chapter 3. Three groups of linear combinations of such ranks are considered,

$$N_2(0,6,3n+i)+N_2(1,6,3n+i)-N_2(2,6,3n+i)-N_2(3,6,3n+i),$$
$$N_2(0,10,5n+d)+N_2(1,10,5n+d)-N_2(4,10,5n+d)-N_2(5,10,5n+d)$$

and

$$N_2(1,10,5n+d)+N_2(2,10,5n+d)-N_2(3,10,5n+d)-N_2(4,10,5n+d),$$

where $0 \leq i \leq 2$ and $0 \leq d \leq 4$. Similar to the results in Chapter 2, the generating functions of the above combinatorial objects are either infinite products or sums of an infinite product and a generalized Lambert series. Inequalities between the M_2-ranks are also discussed.

Our results in Chapter 2 and Chapter 3 give new examples of rank differences. In the previous works by Atkin, Swinnerton-Dyer, Lovejoy and Osburn, they all considered rank differences modulo odd integers. However, the rank differences in this thesis are modulo even integers. Although the general ideas of our proofs are quite similar to Atkin and Swinnerton-Dyer's, the techniques we used are different from the previous works. We hope to provide more examples of rank differences modulo even integers using the methods in Chapter 2 and Chapter 3.

Ramanujan's congruences not only motivated the discoveries of the rank and crank of partitions but also inspired the search for congruences of other special functions. In Chapter 4, we obtain some congruences of the Appell-Lerch sums. Recall that Appell-Lerch sums were first studied by M. P. Appell [9–11] and then by M. Lerch [42]. Following the definition given by D. Hickerson and E. Mortenson in [39, Definition

1.1] and [39, Eq. (3.1)], an Appell-Lerch sum is a series of the form

$$AL(x,q,z) := \frac{1}{(z,q/z,q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r+1} q^{r(r+1)/2} z^{r+1}}{1-q^r xz},$$

where x and z are nonzero complex numbers such that neither z nor xz is an integral power of q. Recently, S. H. Chan studied a special case of the Appell-Lerch sums. He discovered a series of congruences for a function studied by Ramanujan (see [23, Theorem 1.1]). On page 3 of his lost notebook [64], Ramanujan defined the function

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(q;q^2)_{n+1}^2}.$$

Following the notation in [23], let $\sum_{n=0}^{\infty} a(n)q^n := \phi(q)$. Then Chan proved the identity for $a(3n+2)$ which is an analogue of (1.0.4) and (1.0.5),

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^6;q^6)_\infty^{15}}{(q;q^6)_\infty^9 (q^2;q^6)_\infty^2 (q^3;q^6)_\infty^3}. \tag{1.0.8}$$

In particular, we see that $a(3n+2) \equiv 0 \pmod 3$. Indeed, by [23, Eq. 1.16], we know that $\phi(q) = AL(1,q^2,q)$. Generalizing the notation of $a(n)$, we make the following definition.

Definition 1.0.1. *For any integers m, j, and p such that $1 \le j \le p-1$, we define the integer $a_{m,j,p}$ by*

$$\sum_{n=0}^{\infty} a_{m,j,p}(n)q^n := -AL(q^{mp}, q^p, q^j).$$

Then Chan proved that $a_{0,j,p}(pn + pj - j^2) \equiv 0 \pmod p$. Identities analogous to (1.0.8) were also discovered by Chan (see [23, Theorem 1.2]). In Chapter 4, by an argument completely analogous to that in [23], we give an extension of Chan's results. Our next result in Chapter 4 is related to the mock theta functions.

In his last letter to Hardy [64, pp. 127-131], Ramanujan recorded a list of functions which he called mock theta functions. He did not define mock theta functions formally but stated that those functions satisfied certain properties. Also, he did not give a proof of his discovery in his letter. Since then, there was much effort devoted to the study of Ramanujan's mock theta functions (see [3, 36, 68], for example). Many of new mock theta functions were also discovered by many authors. Recently, S. Zwegers opened a new chapter on the study of mock theta functions. In [70, 71], he related Ramanujan's mock theta functions to harmonic weak Maass forms. With the theory of Maass forms and modular forms, striking results were obtained (see [16,31], for example). In particular, the existence of infinitely many congruences for certain mock theta functions were proved. In Chapter 4, using the relations between mock theta functions and Appell-Lerch sums instead of the results by Zwegers, we construct congruences for several mock theta functions. Since, by the results in [39], we know that mock theta functions could always be written as certain linear combinations of Appell-Lerch sums, our congruences follows easily from our identities for Appell-Lerch sums. Although there are a lot of studies on congruences for mock theta functions, to the author's knowledge, Ramanujan type identities (identities like (1.0.4) and (1.0.5)) for such functions have not been discovered before. The identities in Corollary 4.2.1, 4.2.3 and 4.2.4 fill this gap.

Congruence properties of partitions without repeated odd parts were studied in [41,62]. In [27], pairs of partitions without repeated odd parts were considered by W. Y. C. Chen and B. L. S. Lin. Note that pairs of partitions were called bipartitions in [27]. Our notations are slightly different from those in [27]. We count partitions pairs (λ, μ) of n, where each partition, λ and μ, does not have repeated odd parts, and the sum of all the parts of λ and μ is n. We simply call them *partition pairs of n without repeated odd parts*. Let $tt(n)$ denote the number of such partition pairs of n. Then analogies of identities (1.0.4) and (1.0.5) for $tt(n)$ were obtained in [27]. For example, Chen and Lin proved that

$$\sum_{n=0}^{\infty} tt(3n+2)q^n = 3\frac{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^6}{(q;q)_\infty^6 (q^4;q^4)_\infty^6}.$$

In particular, we see that $tt(3n+2) \equiv 0 \pmod 3$. Combinatorial interpretations for the congruence were also found in [27]. Chen and Lin discussed two different ranks for the partitions pairs which were called "biranks". The first one was due to P. Hammond and R. Lewis in [38]. For the partitions pairs (λ, μ), they defined the birank $b(\lambda, \mu)$ by

$$b(\lambda, \mu) = n(\lambda) - n(\mu),$$

where $n(\pi)$ denotes the number of parts of a partition π. Chen and Lin defined another birank $c(\lambda, \mu)$ by

$$c(\lambda, \mu) = l(\lambda) - l(\mu),$$

where $l(\pi)$ denotes the largest part of a partition π. Let $R(t, l, n)$ (resp. $R_2(t, l, n)$) denote the number of partitions pairs (λ, μ) of n without repeated odd parts such that $b(\lambda, \mu)$ (resp. $c(\lambda, \mu)$) $\equiv t \pmod l$. Then by [27, Theorem 5.3, Theorem 5.4], for $0 \leq t \leq 2$, we have

$$R(t, 3, 3n+2) = R_2(t, 3, 3n+2) = \frac{tt(3n+2)}{3}. \tag{1.0.9}$$

This implies $tt(3n+2) \equiv 0 \pmod 3$. In Chapter 5, we give another combinatorial explanation of this congruence. To do this, we make the following definition.

Definition 1.0.2. *The crank of a partition pair without repeated odd parts, (λ, μ), is*

$$\left\lceil \frac{l(\lambda,\mu)}{2} \right\rceil - n(\lambda) - n_o(\mu),$$

where $\lceil\cdot\rceil$ is the ceiling function and $n_o(\mu)$ is the number of odd parts in μ.

We prove that the two-variable generating function of our crank is an infinite product. With this result, an analog of equation (1.0.9) is easily obtained. Two identities of rank-type generating functions are also discovered in Chapter 5. These identities are related to the overpartition pairs. One of them gives a generalization of an identity due to Lovejoy in [53], which was used to derive congruences for overpartition pairs in [54].

Chapter 2

Ranks of partitions modulo 10

2.1 Introduction

As mentioned in Chapter 1, we denote by $N(m,n)$ the number of partitions of n with rank m. The generating function for $N(m,n)$ is given by, [32, Eq. (7.2)],

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}. \tag{2.1.1}$$

Besides (1.0.6) and (1.0.7), more relations between ranks of partitions modulo 5 and 7 have been obtained. As examples, we have

$$N(1,5,5n) > N(2,5,5n) \qquad \text{for } n \geq 1, \tag{2.1.2}$$

$$N(1,5,5n+1) = N(2,5,5n+1) \qquad \text{for } n \geq 0, \tag{2.1.3}$$

$$N(0,5,5n+1) > N(1,5,5n+1) \qquad \text{for } n \geq 0, \tag{2.1.4}$$

$$N(0,5,5n+1) > N(2,5,5n+1) \qquad \text{for } n \geq 0, \tag{2.1.5}$$

$$N(0,5,5n+2) = N(2,5,5n+2) \qquad \text{for } n \geq 0, \tag{2.1.6}$$

$$N(1,5,5n+2) \geq N(2,5,5n+2) \qquad \text{for } n \geq 0, \tag{2.1.7}$$

$$N(1,5,5n+2) \geq N(0,5,5n+2) \quad \text{for } n \geq 0. \tag{2.1.8}$$

Equalities (2.1.3) and (2.1.6) follow from equalities (6.14) and (6.15) in [12], respectively. Garvan proved (2.1.2), (2.1.4) and (2.1.7) in [32]. Inequality (2.1.5) follows from (2.1.3) and (2.1.4), while (2.1.8) follows from (2.1.6) and (2.1.7).

The first objective of this chapter is to prove the following generating functions fof ranks of partitions modulo 10 :

Theorem 2.1.1. *We have*

$$\begin{aligned}
&\sum_{n\geq 0}(N(0,10,n)+N(1,10,n)-N(4,10,n)-N(5,10,n))q^n \\
=&\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{5n}} \qquad (2.1.9)\\
=&\frac{J_{25}J_{20,50}^2J_{50}^5}{J_{10,50}^4J_{15,50}^3}+\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}} \\
&+\frac{qJ_{25}J_{50}^5}{J_{5,50}J_{10,50}^2J_{15,50}^2} \\
&+\frac{q^2J_{25}J_{50}^5}{J_{5,50}^2J_{15,50}J_{20,50}^2} \\
&+\frac{q^3J_{25}J_{10,50}^2J_{50}^5}{J_{5,50}^3J_{20,50}^4}-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+8}}{1+q^{25n+10}} \\
&+\frac{2q^4J_{50}^6}{J_{25}J_{5,50}J_{10,50}J_{15,50}J_{20,50}} \qquad (2.1.10)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n\geq 0}(N(1,10,n)+N(2,10,n)-N(3,10,n)-N(4,10,n))q^n \\
=&\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}} \qquad (2.1.11)\\
=&\frac{2q^5J_{50}^6}{J_{25}J_{10,50}^2J_{15,50}^2}-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}}
\end{aligned}$$

$$+\frac{2q^6J_{50}^6}{J_{25}J_{5,50}J_{15,50}J_{20,50}^2}$$
$$+\frac{q^2J_{25}J_{20,50}J_{50}^5}{J_{10,50}^3J_{15,50}^3}$$
$$+\frac{q^3J_{25}J_{50}^5}{J_{5,50}J_{10,50}J_{15,50}^2J_{20,50}}$$
$$+\frac{J_{25}J_{20,50}^2J_{25,50}J_{50}^5}{2qJ_{10,50}^4J_{15,50}^4}-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(75n^2+25n)/2-1}}{1+q^{25n}}. \tag{2.1.12}$$

With the generating functions for $N(0,10,5n+l)+N(1,10,5n+l)-N(4,10,5n+l)-N(5,10,5n+l)$ and $N(1,10,5n+l)+N(2,10,5n+l)-N(3,10,5n+l)-N(4,10,5n+l)$ $(0\leq l\leq 4)$ established in Theorem 2.1.1, we discover the following inequalities between the ranks of partitions modulo 10.

Corollary 2.1.2. *For $n\geq 0$, $i=1,2,3,4$ and $j=1,2,3,4$, we have*

$$N(0,10,5n+i)+N(1,10,5n+i)>N(4,10,5n+i)+N(5,10,5n+i), \tag{2.1.13}$$
$$N(1,10,5n+j)+N(2,10,5n+j)\geq N(3,10,5n+j)+N(4,10,5n+j), \tag{2.1.14}$$
$$N(0,10,5n+1)>N(4,10,5n+1), \tag{2.1.15}$$
$$N(1,10,5n+1)>N(5,10,5n+1), \tag{2.1.16}$$
$$N(1,10,5n+1)\geq N(3,10,5n+1), \tag{2.1.17}$$
$$N(2,10,5n+1)\geq N(4,10,5n+1), \tag{2.1.18}$$
$$N(1,10,5n+2)>N(5,10,5n+2), \tag{2.1.19}$$
$$N(1,10,5n+2)\geq N(3,10,5n+2), \tag{2.1.20}$$
$$N(0,10,5n+4)>N(4,10,5n+4), \tag{2.1.21}$$
$$N(1,10,5n+4)>N(5,10,5n+4), \tag{2.1.22}$$
$$N(2,10,5n+4)\geq N(4,10,5n+4), \tag{2.1.23}$$

$$N(1,10,5n+4) \geq N(3,10,5n+4). \tag{2.1.24}$$

For (2.1.14)*,* (2.1.17)*,* (2.1.18)*,* (2.1.20)*,* (2.1.23) *and* (2.1.24)*, we have strict inequalities except in the following cases:* $(j,n) = (1,0), (2,1)$ *and* $(4,0)$ *in* (2.1.14)*,* $n = 0$ *in* (2.1.17)*,* (2.1.18)*,* (2.1.23) *and* (2.1.24)*,* $n = 1$ *in* (2.1.20)*.*

Remark:

1. Relations between ranks of partitions were also discussed by Lewis and N. Santa-Gadea (see [44, 47, 65], for example). Numerous relations between ranks and cranks of partitions were found over the years (see [8, 43, 45, 46, 48, 49], for example). Relations between cranks of partitions modulo 10 are also discussed in [33].

2. Computer evidence suggests that (2.1.13) and (2.1.14) might still hold when $i = j = 0$. We failed to prove them and so we leave them as the following conjecture.

Conjecture 2.1.3.

$$N(0,10,5n) + N(1,10,5n) > N(4,10,5n) + N(5,10,5n) \quad \textit{for } n \geq 0,$$

$$N(1,10,5n) + N(2,10,5n) \geq N(3,10,5n) + N(4,10,5n) \quad \textit{for } n \geq 1.$$

The chapter is organized as follows. We first give some lemmas in Section 2.2, then prove Theorem 2.1.1 in Section 2.3. In Section 2.4 we prove the inequalities.

2.2 Some lemmas

First, we derive some theta function identities used repeatedly in the proof of Theorem 2.1.1. Let

$$H(a,b,c,q) := \frac{[ab,bc,ca;q]_\infty (q;q)_\infty^2}{[a,b,c,abc;q]_\infty}. \tag{2.2.1}$$

We have the following equivalent version of Corollary 4.4 in [23]:

Lemma 2.2.1.

$$H(a,b,c,q) - H(a,b,d,q) = H(c,1/d,abd,q), \tag{2.2.2}$$

$$H(a,a,q^{25}/a,q^{50}) + H(b,b,q^{25}/b,q^{50}) = 2H(a,q^{25}/a,b,q^{50}), \tag{2.2.3}$$

$$H(a,a,q^{25}/a,q^{50}) - H(b,b,q^{25}/b,q^{50}) = 2H(a,q^{25}/a,b/q^{25},q^{50}). \tag{2.2.4}$$

We need the following special cases. First, replacing q by q^{50} and setting $(a,b,c,d) = (q^{10},q^{25},q^{10},q^{-5})$ in (2.2.2), we find that

$$\frac{J_{15,50}^2 J_{20,50}}{J_{5,50}J_{10,50}^2 J_{25,50}} + \frac{q^5 J_{15,50}}{J_{10,50}J_{25,50}} = \frac{J_{15,50}^2}{J_{5,50}^2 J_{20,50}}. \tag{2.2.5}$$

Replacing q by q^{50} and setting $(a,b,c,d) = (q^{15},q^{15},q^{10},q^{-10})$ in (2.2.2), we find that

$$\frac{J_{20,50}J_{25,50}^2}{J_{10,50}^2 J_{15,50}^2} + \frac{q^{10} J_{5,50}^2}{J_{10,50}J_{15,50}^2} = \frac{J_{20,50}^2}{J_{10,50}^3}.$$

Multiplying by $\frac{J_{10,50}J_{50}^6}{J_{5,50}^2 J_{20,50}^2}$ throughout and rearranging, the above equation becomes

$$\frac{J_{25,50}^2 J_{50}^6}{J_{5,50}^2 J_{10,50}J_{15,50}^2 J_{20,50}} = \frac{J_{50}^6}{J_{5,50}^2 J_{10,50}^2} - \frac{q^{10} J_{50}^6}{J_{15,50}^2 J_{20,50}^2}. \tag{2.2.6}$$

Replacing q by q^{50} and setting $(a,b,c,d) = (q^{15},q^{10},q^{10},q^{-5})$ in (2.2.2), we find that

$$\frac{J_{20,50}J_{25,50}^2}{J_{10,50}^2 J_{15,50}^2} + \frac{q^5 J_{25,50}}{J_{15,50}J_{20,50}} = \frac{J_{25,50}}{J_{5,50}J_{10,50}}. \tag{2.2.7}$$

Next, replacing q by q^{25} and setting $(a,b,c,d) = (-1,-q^{10},-q^5,q^5)$, in (2.2.2), we find that

$$\frac{J_{5,25}J_{10,25}}{\overline{J}_{5,25}\overline{J}_{10,25}} - \frac{\overline{J}_{5,25}\overline{J}_{10,25}}{J_{5,25}J_{10,25}} = -\frac{4q^5\overline{J}_{25}^4J_{25}^2}{J_{5,25}J_{10,25}}. \tag{2.2.8}$$

Replacing q by q^{25} and setting $(a,b,c,d) = (-q^5,-q^5,-q^5,q^5)$ in (2.2.2), we find that

$$\frac{J_{10,25}^2}{\overline{J}_{5,25}\overline{J}_{10,25}} - \frac{\overline{J}_{10,25}^2}{J_{5,25}J_{10,25}} = -\frac{2q^5\overline{J}_{25}^2\overline{J}_{5,25}^2J_{25}}{J_{5,25}J_{10,25}\overline{J}_{10,25}}. \tag{2.2.9}$$

Setting $a = q^5, b = q^{10}$ in (2.2.3) and (2.2.4), respectively, we find that

$$\frac{J_{10,50}}{J_{5,50}^2J_{20,50}^2} + \frac{J_{20,50}}{J_{10,50}^2J_{15,50}^2} = \frac{2}{J_{5,50}J_{10,50}J_{25,50}} \tag{2.2.10}$$

and

$$\frac{J_{10,50}}{J_{5,50}^2J_{20,50}^2} - \frac{J_{20,50}}{J_{10,50}^2J_{15,50}^2} = \frac{2q^5}{J_{15,50}J_{20,50}J_{25,50}}. \tag{2.2.11}$$

Then multiplying (2.2.11) by (2.2.10), we get

$$\frac{J_{10,50}^2}{J_{5,50}^4J_{20,50}^4} - \frac{J_{20,50}^2}{J_{10,50}^4J_{15,50}^4} = \frac{4q^5}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}J_{25,50}^2}. \tag{2.2.12}$$

We also need [32, Lemma (3.18)] as follows:

Lemma 2.2.2.

$$(q;q)_\infty = J_{25} \times \left\{\frac{J_{10,25}}{J_{5,25}} - q - q^2\frac{J_{5,25}}{J_{10,25}}\right\}. \tag{2.2.13}$$

The following lemma is a special case with $r = 0, s = 3$ of Theorem 2.1 in [21].

Lemma 2.2.3. *For* $|q| < |\frac{1}{b_1b_2b_3}| < 1$,

$$\frac{(q;q)_\infty^2}{[b_1,b_2,b_3;q]_\infty} = \frac{1}{[b_2/b_1,b_3/b_1;q]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{1-b_1q^n}\left(\frac{b_1^2}{b_2b_3}\right)^n$$

$$+\frac{1}{[b_1/b_2, b_3/b_2; q]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-b_2q^n}\left(\frac{b_2^2}{b_1b_3}\right)^n$$
$$+\frac{1}{[b_1/b_3, b_2/b_3; q]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-b_3q^n}\left(\frac{b_3^2}{b_2b_2}\right)^n. \qquad (2.2.14)$$

We also require a result (see [50, Theorem 1]) on nonnegativity of coefficients of certain products.

Lemma 2.2.4. *If p and r be positive integers with $p \geq 2$ and $r < p$ and*

$$L_{p,r}(q) := \sum_{n=0}^{\infty} b_{p,r}(n)q^n := \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty(q^{p-r};q^p)_\infty},$$

then $b_{p,r}(n) \geq 0$ for all n. Moreover, if we let

$$L_{p,r}(q) + q^p =: \sum_{n=0}^{\infty} c_{p,r}(n)q^n =: \sum_0 + \sum_1 + \cdots + \sum_{r-1}, \qquad (2.2.15)$$

where

$$\sum_i = \sum_{n=0}^{\infty} c_{p,r}(nr+i)q^{nr+i},$$

then for each $0 \leq i \leq r-1$, the sequence $\{c_{p,r}(nr+i)\}_{n\geq 0}$ is non-decreasing.

2.3 Proof of Theorem 2.1.1

The following lemma is a key to the proof of Theorem 2.1.1. It is used to obtain (2.3.42) and (2.3.43). The infinite sums on the left hand sides of the equations in Lemma 2.3.1 come from (2.1.9) and (2.1.11).

Lemma 2.3.1. *Let*

$$P(a,b) := \frac{[a, a^2; q^{25}]_\infty (q^{25};q^{25})_\infty^2}{[b/a, ab, b; q^{25}]_\infty}.$$

We have

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}}=P(q^5,-q^5)-\frac{P(q^{10},-q^5)}{q^3}$$
$$+\frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}}, \tag{2.3.1}$$
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2+n}}{1+q^{5n}}=P(q^{10},-q^{10})-q^3P(q^5,-q^{10})$$
$$-\frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+8}}{1+q^{25n+10}}, \tag{2.3.2}$$
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2+2n}}{1+q^{5n}}=\frac{P(q^5,-1)}{q^6}-\frac{P(q^{10},-1)}{q^9}$$
$$-\frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+25n)/2-1}}{1+q^{25n}}. \tag{2.3.3}$$

Proof. Since the proofs of the above three equations are similar, we only show (2.3.1). First, replacing n with $-n$ in the summation index of the series on the left side of (2.3.1), we find that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+9)/2}}{1+q^{5n}}. \tag{2.3.4}$$

Splitting the series on the right side of (2.3.4) into five series according to the summation index n modulo 5, we find that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}}$$
$$=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+9)/2}}{1+q^{5n}}$$
$$=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+45n)/2}}{1+q^{25n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+75n)/2+6}}{1+q^{25n+5}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+105n)/2+15}}{1+q^{25n+10}}$$
$$-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+135n)/2+27}}{1+q^{25n+15}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+165n)/2+45}}{1+q^{25n+20}}$$

$$=: T_0 - T_1 + T_2 - T_3 + T_4. \tag{2.3.5}$$

We apply identity (2.2.14) twice. First, replacing q, b_1, b_2 and b_3 by $q^{25}, -1, -q^{10}$ and $-q^5$, respectively, in (2.2.14), we find that

$$P(q^5, -q^5) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+45n)/2}}{1+q^{25n}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+105n)/2+15}}{1+q^{25n+10}} - \frac{J_{10,25}}{J_{5,25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+75n)/2+5}}{1+q^{25n+5}},$$

which is equivalent to

$$T_0 + T_2 = P(q^5, -q^5) + \frac{J_{10,25}}{qJ_{5,25}} T_1. \tag{2.3.6}$$

Next, replacing q, b_1, b_2 and b_3 by $q^{25}, -q^{15}, -q^{-5}$ and $-q^5$, respectively, in (2.2.14), we find that

$$P(q^{10}, -q^5) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+135n)/2+30}}{1+q^{25n+15}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+15n)/2}}{1+q^{25n-5}} - \frac{J_{5,25}}{J_{10,25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+75n)/2+10}}{1+q^{25n+5}},$$

which is equivalent to

$$T_3 - T_4 = \frac{P(q^{10}, -q^5)}{q^3} + \frac{qJ_{5,25}}{J_{10,25}} T_1. \tag{2.3.7}$$

Substituting (2.3.6) and (2.3.7) into (2.3.5), we find that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}} &= P(q^5, -q^5) - \frac{P(q^{10}, -q^5)}{q^3} + \frac{J_{10,25}}{qJ_{5,25}} T_1 - T_1 - \frac{qJ_{5,25}}{J_{10,25}} T_1 \\ &= P(q^5, -q^5) - \frac{P(q^{10}, -q^5)}{q^3} + \left\{ \frac{J_{10,25}}{J_{5,25}} - q - \frac{q^2 J_{5,25}}{J_{10,25}} \right\} \frac{T_1}{q}. \end{aligned}$$

We complete the proof after invoking (2.2.13) on the right side of the last equality. □

Next, we obtain the 5-dissection of

$$\frac{1}{(q;q)_\infty}\left\{P(q^5,-q^5)-\frac{P(q^{10},-q^5)}{q^3}+P(q^{10},-q^{10})-q^3P(q^5,-q^{10})\right\}$$

and

$$\frac{1}{(q;q)_\infty}\left\{\frac{P(q^{10},-q^5)}{q^3}-P(q^5,-q^5)+\frac{P(q^5,-1)}{q^6}-\frac{P(q^{10},-1)}{q^9}\right\}$$

via the next two lemmas which give us all the infinite products in Theorem 2.1.1. Lemma 2.3.2 (resp. Lemma 2.3.3) is applied in the final step of the proof of (2.1.10) (resp. (2.1.12)).

Lemma 2.3.2. *Let*

$$\begin{aligned}
A_0 &:= \frac{(q^{25};q^{50})_\infty J_{50}^6 J_{20,50}^2}{J_{10,50}^4 J_{15,50}^3}\\
A_1 &:= \frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2 J_{15,50}^2}\\
A_2 &:= \frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^2}\\
A_3 &:= \frac{q^3(q^{25};q^{50})_\infty J_{50}^6 J_{10,50}^2}{J_{5,50}^3 J_{20,50}^4}\\
A_4 &:= \frac{2q^4 J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}}.
\end{aligned}$$

We have

$$\begin{aligned}
&P(q^5,-q^5)-\frac{P(q^{10},-q^5)}{q^3}+P(q^{10},-q^{10})-q^3P(q^5,-q^{10})\\
&=(q;q)_\infty\times\{A_0+A_1+A_2+A_3+A_4\}. \qquad (2.3.8)
\end{aligned}$$

Proof. By Lemma 2.2.13, we have

$$(q;q)_\infty=\frac{J_{10,25}J_{25}}{J_{5,25}}-qJ_{25}-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}.$$

Expanding the right side of (2.3.8) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the five identities:

$$
\begin{aligned}
&P(q^5,-q^5)+P(q^{10},-q^{10})\\
&=\frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2}{J_{10,50}^4J_{15,50}^3}\times\frac{J_{10,25}J_{25}}{J_{5,25}}-qJ_{25}\times\frac{2q^4J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}}\\
&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{q^3(q^{25};q^{50})_\infty J_{50}^6J_{10,50}^2}{J_{5,50}^3J_{20,50}^4},
\end{aligned}
\tag{2.3.9}
$$

$$
\begin{aligned}
0&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}-qJ_{25}\times\frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2}{J_{10,50}^4J_{15,50}^3}\\
&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{2q^4J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}},
\end{aligned}
\tag{2.3.10}
$$

$$
\begin{aligned}
-\frac{P(q^{10},-q^5)}{q^3}&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}-qJ_{25}\times\frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}\\
&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2}{J_{10,50}^4J_{15,50}^3},
\end{aligned}
\tag{2.3.11}
$$

$$
\begin{aligned}
-P(q^5,-q^{10})q^3&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{q^3(q^{25};q^{50})_\infty J_{50}^6J_{10,50}^2}{J_{5,50}^3J_{20,50}^4}-qJ_{25}\times\frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}\\
&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2},
\end{aligned}
\tag{2.3.12}
$$

$$
\begin{aligned}
0&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{2q^4J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}}-qJ_{25}\times\frac{q^3(q^{25};q^{50})_\infty J_{50}^6J_{10,50}^2}{J_{5,50}^3J_{20,50}^4}\\
&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}.
\end{aligned}
\tag{2.3.13}
$$

Simplifying each of these five identities and noting that

$$P(q^5,-q^5)=P(q^{10},-q^{10})=\frac{J_{5,50}^2J_{10,50}J_{15,50}^2J_{20,50}J_{25,50}^2}{2J_{50}^6},$$

we see that to prove (2.3.9), it suffices to show that

$$
\begin{aligned}
&\frac{J_{5,50}^2J_{10,50}J_{15,50}^2J_{20,50}J_{25,50}^2}{J_{50}^6}\\
&=\frac{J_{20,50}J_{25,50}J_{50}^6}{J_{5,50}J_{10,50}^3J_{15,50}^2}-\frac{2q^5J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}}-\frac{q^5J_{10,50}J_{25,50}J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^3}.
\end{aligned}
\tag{2.3.14}
$$

Multiplying by $\frac{q^5 J_{10,50} J_{25,50} J_{50}^6}{J_{15,50}^3 J_{20,50}^2}$ on both sides of (2.2.5) and rearranging, we have

$$\frac{q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}} - \frac{q^5 J_{10,50} J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^3} = -\frac{q^{10} J_{50}^6}{J_{15,50}^2 J_{20,50}^2}. \tag{2.3.15}$$

Next, multiplying by $\frac{J_{50}^6}{J_{5,50} J_{10,50} J_{25,50}}$ on both sides of (2.2.7), we obtain

$$\frac{J_{20,50} J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^3 J_{15,50}^2} + \frac{q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}} = \frac{J_{50}^6}{J_{5,50}^2 J_{10,50}^2}. \tag{2.3.16}$$

Adding (2.3.15) to (2.3.16), we find that

$$\begin{aligned} &\frac{2q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}} - \frac{q^5 J_{10,50} J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^3} + \frac{J_{20,50} J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^3 J_{15,50}^2} \\ &= \frac{J_{50}^6}{J_{5,50}^2 J_{10,50}^2} - \frac{q^{10} J_{50}^6}{J_{15,50}^2 J_{20,50}^2} \\ &= \frac{J_{25,50}^2 J_{50}^6}{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}}, \end{aligned} \tag{2.3.17}$$

where the second equality follows from (2.2.6). Adding (2.3.17) to (2.3.14) and simplifying, we find that (2.3.14) is equivalent to

$$\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50} J_{25,50}^2}{J_{50}^6} = \frac{J_{25,50}^2 J_{50}^6}{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}} - \frac{4q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}}. \tag{2.3.18}$$

Dividing both sides by $J_{25,50}^2$ and noting that

$$\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}}{J_{50}^6} = \frac{J_{5,25} J_{10,25}}{\overline{J}_{5,25} \overline{J}_{10,25}},$$
$$\frac{J_{50}^6}{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}} = \frac{\overline{J}_{5,25} \overline{J}_{10,25}}{J_{5,25} J_{10,25}}$$

and

$$\frac{4q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50} J_{25,50}^2} = \frac{4q^5 \overline{J}_{25}^4 J_{25}^2}{J_{5,25} J_{10,25}},$$

we find that (2.3.18) is equivalent to

$$\frac{J_{5,25} J_{10,25}}{\overline{J}_{5,25} \overline{J}_{10,25}} = \frac{\overline{J}_{5,25} \overline{J}_{10,25}}{J_{5,25} J_{10,25}} - \frac{4q^5 \overline{J}_{25}^4 J_{25}^2}{J_{5,25} J_{10,25}}.$$

This is precisely (2.2.8), completing the proof of (2.3.9).

To prove (2.3.10), it suffices to show that

$$0=\frac{J_{25,50}J_{50}^6}{J_{5,50}^2J_{10,50}J_{15,50}J_{20,50}}-\frac{J_{20,50}^2J_{25,50}J_{50}^6}{J_{10,50}^4J_{15,50}^3}-\frac{2q^5J_{50}^6}{J_{10,50}^2J_{15,50}^2},$$

which follows from multiplying $\frac{J_{20,50}J_{25,50}J_{50}^6}{J_{10,50}^2J_{15,50}}$ on both sides of (2.2.11).

To prove (2.3.11), it suffices to show that

$$\begin{aligned}&-\frac{J_{5,50}^3J_{15,50}^2J_{20,50}^2J_{25,50}}{J_{50}^6}\\&=\frac{J_{10,50}J_{25,50}J_{50}^6}{J_{5,50}^3J_{20,50}^3}-\frac{J_{25,50}J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}-\frac{J_{5,50}J_{20,50}^3J_{25,50}J_{50}^6}{J_{10,50}^5J_{15,50}^4}.\end{aligned}\tag{2.3.19}$$

Multiplying $\frac{J_{25,50}J_{50}^6}{J_{5,50}J_{20,50}}$ on both sides of (2.2.11), we have

$$\frac{J_{10,50}J_{25,50}J_{50}^6}{J_{5,50}^3J_{20,50}^3}-\frac{J_{25,50}J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}=\frac{2q^5J_{50}^6}{J_{5,50}J_{15,50}J_{20,50}^2}.$$

Substituting the above identity into (2.3.19), we find that (2.3.19) is equivalent to

$$-\frac{J_{5,50}^3J_{15,50}^2J_{20,50}^2J_{25,50}}{J_{50}^6}=\frac{2q^5J_{50}^6}{J_{5,50}J_{15,50}J_{20,50}^2}-\frac{J_{5,50}J_{20,50}^3J_{25,50}J_{50}^6}{J_{10,50}^5J_{15,50}^4}.\tag{2.3.20}$$

Multiplying by $\frac{J_{10,50}}{J_{5,50}J_{25,50}J_{50}}$ and noting that

$$\begin{aligned}\frac{J_{5,50}^2J_{10,50}J_{15,50}^2J_{20,50}^2}{J_{50}^7}&=\frac{J_{5,25}J_{10,25}^2}{\overline{J}_{5,25}J_{25}^2},\\ \frac{2q^5J_{10,50}J_{50}^5}{J_{5,50}^2J_{15,50}J_{20,50}^2J_{25,50}}&=\frac{2q^5\overline{J}_{5,25}^2\overline{J}_{25}^2}{J_{10,25}J_{25}}\end{aligned}$$

and

$$\frac{J_{20,50}^3J_{50}^5}{J_{10,50}^4J_{15,50}^4}=\frac{\overline{J}_{10,25}^3}{J_{10,25}J_{25}^2},$$

we find that (2.3.20) is equivalent to

$$-\frac{J_{5,25}J_{10,25}^2}{\overline{J}_{5,25}J_{25}^2}=\frac{2q^5\overline{J}_{5,25}^2\overline{J}_{25}^2}{J_{10,25}J_{25}}-\frac{\overline{J}_{10,25}^3}{J_{10,25}J_{25}^2}.$$

This follows from multiplying $\frac{J_{5,25}\overline{J}_{10,25}}{J_{25}^2}$ on both sides of (2.2.9). So the proof of (2.3.11) is completed.

To prove (2.3.12), it suffices to show that

$$\begin{aligned}&-\frac{J_{5,50}^2J_{10,50}^2J_{15,50}^3J_{25,50}}{J_{50}^6}\\&=\frac{J_{10,50}^3J_{15,50}J_{25,50}J_{50}^6}{J_{5,50}^4J_{20,50}^5}-\frac{J_{25,50}J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}-\frac{J_{20,50}J_{25,50}J_{50}^6}{J_{10,50}^3J_{15,50}^3}.\end{aligned} \tag{2.3.21}$$

Multiplying $\frac{J_{10,50}^2J_{15,50}J_{25,50}J_{50}^6}{J_{5,50}^2J_{20,50}^3}$ on both sides of (2.2.11), we find that

$$\frac{J_{10,50}^3J_{15,50}J_{25,50}J_{50}^6}{J_{5,50}^4J_{20,50}^5}-\frac{J_{25,50}J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}=\frac{2q^5J_{10,50}^2J_{50}^6}{J_{5,50}^2J_{20,50}^4}.$$

Substituting the above identity into (2.3.21), we find that (2.3.21) is equivalent to

$$-\frac{J_{5,50}^2J_{10,50}^2J_{15,50}^3J_{25,50}}{J_{50}^6}=\frac{2q^5J_{10,50}^2J_{50}^6}{J_{5,50}^2J_{20,50}^4}-\frac{J_{20,50}J_{25,50}J_{50}^6}{J_{10,50}^3J_{15,50}^3}. \tag{2.3.22}$$

This follows from multiplying $\frac{J_{10,50}^2J_{15,50}}{J_{5,50}J_{20,50}^2}$ on both sides of (2.3.20). This completes the proof of (2.3.12).

To prove (2.3.13), it suffices to show that

$$0=\frac{2J_{50}^6}{J_{5,50}^2J_{20,50}^2}-\frac{J_{10,50}^2J_{25,50}J_{50}^6}{J_{5,50}^3J_{20,50}^4}-\frac{J_{25,50}J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}^2J_{20,50}},$$

which follows from multiplying $\frac{J_{10,50}J_{25,50}J_{50}^6}{J_{5,50}J_{20,50}^2}$ on both sides of (2.2.10). □

Lemma 2.3.3. *Let*

$$\begin{aligned}B_0&:=\frac{2q^5J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2J_{15,50}^2}\\B_1&:=\frac{2q^6J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{15,50}J_{20,50}^2}\\B_2&:=\frac{q^2(q^{25};q^{50})_\infty J_{50}^6J_{20,50}}{J_{10,50}^3J_{15,50}^3}\end{aligned}$$

$$B_3 := \frac{q^3(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}^2J_{20,50}}$$

$$B_4 := \frac{(q^{25};q^{50})_\infty J_{50}^6 J_{20,50}^2 J_{25,50}}{2qJ_{10,50}^4 J_{15,50}^4}.$$

We have

$$\frac{P(q^{10},-q^5)}{q^3} - P(q^5,-q^5) + \frac{P(q^5,-1)}{q^6} - \frac{P(q^{10},-1)}{q^9}$$
$$= (q;q)_\infty \times \{B_0 + B_1 + B_2 + B_3 + B_4\}. \tag{2.3.23}$$

Proof. Expanding the right side of (2.3.23) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the five identities,

$$-P(q^5,-q^5) = \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{2q^5J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2J_{15,50}^2} - qJ_{25} \times \frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2J_{25,50}}{2qJ_{10,50}^4J_{15,50}^4}$$
$$- \frac{q^2J_{5,25}J_{25}}{J_{10,25}} \times \frac{q^3(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}^2J_{20,50}}, \tag{2.3.24}$$

$$-\frac{P(q^{10},-1)}{q^9} = \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{2q^6J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{15,50}J_{20,50}^2} - qJ_{25} \times \frac{2q^5J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2J_{15,50}^2}$$
$$- \frac{q^2J_{5,25}J_{25}}{J_{10,25}} \times \frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2J_{25,50}}{2qJ_{10,50}^4J_{15,50}^4}, \tag{2.3.25}$$

$$\frac{P(q^{10},-q^5)}{q^3} = \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{q^2(q^{25};q^{50})_\infty J_{50}^6J_{20,50}}{J_{10,50}^3J_{15,50}^3} - qJ_{25} \times \frac{2q^6J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{15,50}J_{20,50}^2}$$
$$- \frac{q^2J_{5,25}J_{25}}{J_{10,25}} \times \frac{2q^5J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2J_{15,50}^2}, \tag{2.3.26}$$

$$0 = \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{q^3(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}^2J_{20,50}} - qJ_{25} \times \frac{q^2(q^{25};q^{50})_\infty J_{50}^6J_{20,50}}{J_{10,50}^3J_{15,50}^3}$$
$$- \frac{q^2J_{5,25}J_{25}}{J_{10,25}} \times \frac{2q^6J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{15,50}J_{20,50}^2}, \tag{2.3.27}$$

$$\frac{P(q^5,-1)}{q^6} = \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2J_{25,50}}{2qJ_{10,50}^4J_{15,50}^4} - qJ_{25} \times \frac{q^3(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}^2J_{20,50}}$$
$$- \frac{q^2J_{5,25}J_{25}}{J_{10,25}} \times \frac{q^2(q^{25};q^{50})_\infty J_{50}^6J_{20,50}}{J_{10,50}^3J_{15,50}^3}. \tag{2.3.28}$$

Simplifying each of these five identities, we see that to prove (2.3.24), it suffices to show that

$$-\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50} J_{25,50}^2}{2J_{50}^6} = \frac{2q^5 J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}} - \frac{J_{50}^6 J_{20,50}^2 J_{25,50}^2}{2J_{10,50}^4 J_{15,50}^4} - \frac{q^5 J_{50}^6 J_{25,50}}{J_{10,50}^2 J_{15,50}^3}. \tag{2.3.29}$$

Multiplying $\frac{q^5 J_{25,50} J_{50}^6}{J_{15,50} J_{20,50}}$ on both sides of (2.2.10) and rearranging, we find that

$$\frac{2q^5 J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}} - \frac{q^5 J_{50}^6 J_{25,50}}{J_{10,50}^2 J_{15,50}^3} = \frac{q^5 J_{50}^6 J_{10,50} J_{25,50}}{J_{5,50}^2 J_{15,50} J_{20,50}^3}.$$

Substituting the above equality into (2.3.29), we find that (2.3.29) is equivalent to

$$-\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50} J_{25,50}^2}{2J_{50}^6} = \frac{q^5 J_{50}^6 J_{10,50} J_{25,50}}{J_{5,50}^2 J_{15,50} J_{20,50}^3} - \frac{J_{50}^6 J_{20,50}^2 J_{25,50}^2}{2J_{10,50}^4 J_{15,50}^4}. \tag{2.3.30}$$

This follows from multiplying $\frac{J_{10,50} J_{25,50}}{2J_{5,50} J_{20,50}}$ on both sides of (2.3.20). This completes the proof of (2.3.24).

To prove (2.3.25), it suffices to show that

$$-\frac{J_{5,50} J_{10,50}^3 J_{15,50}^3 J_{25,50}^2}{2J_{50}^6 J_{20,50}} = \frac{2q^5 J_{10,50} J_{50}^6}{J_{5,50}^2 J_{20,50}^3} - \frac{2q^5 J_{50}^6}{J_{10,50}^2 J_{15,50}^2} - \frac{J_{5,50} J_{20,50}^3 J_{25,50}^2 J_{50}^6}{2J_{10,50}^5 J_{15,50}^5}. \tag{2.3.31}$$

Multiplying $\frac{J_{5,50} J_{20,50} J_{25,50}^2 J_{50}^6}{2J_{10,50} J_{15,50}}$ on both sides of (2.2.12) and rearranging, we find that

$$-\frac{2q^5 J_{50}^6}{J_{10,50}^2 J_{15,50}^2} - \frac{J_{5,50} J_{20,50}^3 J_{25,50}^2 J_{50}^6}{2J_{10,50}^5 J_{15,50}^5} = -\frac{J_{10,50} J_{25,50}^2 J_{50}^6}{2J_{5,50}^3 J_{15,50} J_{20,50}^3}.$$

Substituting the above equality into (2.3.31), we find that (2.3.31) is equivalent to

$$-\frac{J_{5,50} J_{10,50}^3 J_{15,50}^3 J_{25,50}^2}{2J_{50}^6 J_{20,50}} = \frac{2q^5 J_{10,50} J_{50}^6}{J_{5,50}^2 J_{20,50}^3} - \frac{J_{10,50} J_{25,50}^2 J_{50}^6}{2J_{5,50}^3 J_{15,50} J_{20,50}^3}.$$

This follows from multiplying $-\frac{J_{10,50}^2 J_{15,50}}{2J_{5,50} J_{20,50}^2}$ on both sides of (2.3.18). This completes the proof of (2.3.25).

To prove (2.3.26), it suffices to show that

$$\frac{J_{5,50}^3 J_{15,50}^2 J_{20,50}^2 J_{25,50}}{J_{50}^6} = \frac{J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^2 J_{15,50}^2} - \frac{2q^5 J_{50}^6}{J_{5,50} J_{15,50} J_{20,50}^2} - \frac{2q^5 J_{5,50} J_{20,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3}. \tag{2.3.32}$$

Multiplying $\frac{J_{5,50} J_{20,50}^2 J_{25,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^2}$ on both sides of (2.2.11) and rearranging, we find that

$$-\frac{2q^5 J_{5,50} J_{20,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3} + \frac{J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^2 J_{15,50}^2} = \frac{J_{5,50} J_{20,50}^3 J_{25,50} J_{50}^6}{J_{10,50}^5 J_{15,50}^4}.$$

Substituting the above equality into (2.3.32), we find that (2.3.32) is equivalent to

$$\frac{J_{5,50}^3 J_{15,50}^2 J_{20,50}^2 J_{25,50}}{J_{50}^6} = \frac{J_{5,50} J_{20,50}^3 J_{25,50} J_{50}^6}{J_{10,50}^5 J_{15,50}^4} - \frac{2q^5 J_{50}^6}{J_{5,50} J_{15,50} J_{20,50}^2}. \tag{2.3.33}$$

This follows from (2.3.20). This completes the proof of (2.3.26).

To prove (2.3.27), it suffices to show that

$$0 = \frac{J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^2} - \frac{J_{20,50} J_{25,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3} - \frac{2q^5 J_{50}^6}{J_{10,50} J_{15,50}^2 J_{20,50}},$$

which follows from multiplying $\frac{J_{25,50} J_{50}^6}{J_{10,50} J_{15,50}}$ on both sides of (2.2.11).

To prove (2.3.28), it suffices to show that

$$\frac{J_{5,50}^3 J_{15,50} J_{20,50}^3 J_{25,50}^2}{2 J_{10,50} J_{50}^6} = \frac{J_{20,50} J_{25,50}^2 J_{50}^6}{2 J_{5,50} J_{10,50}^3 J_{15,50}^3} - \frac{q^5 J_{25,50} J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}} - \frac{q^5 J_{5,50} J_{20,50}^2 J_{25,50} J_{50}^6}{J_{10,50}^4 J_{15,50}^4}. \tag{2.3.34}$$

Multiplying $\frac{J_{5,50} J_{20,50}^3 J_{25,50}^2 J_{50}^6}{2 J_{10,50}^4 J_{15,50}^3}$ on both sides of (2.2.11) and rearranging, we find that

$$\frac{J_{20,50} J_{25,50}^2 J_{50}^6}{2 J_{5,50} J_{10,50}^3 J_{15,50}^3} - \frac{q^5 J_{5,50} J_{20,50}^2 J_{25,50} J_{50}^6}{J_{10,50}^4 J_{15,50}^4} = \frac{J_{5,50} J_{20,50}^4 J_{25,50}^2 J_{50}^6}{2 J_{10,50}^6 J_{15,50}^5}.$$

Substituting the above equality into (2.3.34), we find that (2.3.34) is equivalent to

$$\frac{J_{5,50}^3 J_{15,50} J_{20,50}^3 J_{25,50}^2}{2 J_{10,50} J_{50}^6} = \frac{J_{5,50} J_{20,50}^4 J_{25,50}^2 J_{50}^6}{2 J_{10,50}^6 J_{15,50}^5} - \frac{q^5 J_{25,50} J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}}. \tag{2.3.35}$$

This follows from multiplying $\frac{J_{20,50} J_{25,50}}{2 J_{10,50} J_{15,50}}$ on both sides of (2.3.20). □

Now we prove Theorem 2.1.1.

Proof. First, we prove (2.1.9) and (2.1.11). Replacing z by $\xi = e^{\frac{\pi i}{5}}$ in (2.1.1), we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\xi q, q/\xi; q)_n} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n)\xi^m q^n = \sum_{n=0}^{\infty} \sum_{m=0}^{9} \sum_{j=-\infty}^{\infty} N(10j+m,n)\xi^m q^n.$$

By the definition of $N(m, 10, n)$, we know that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\xi q, q/\xi; q)_n} = \sum_{n=0}^{\infty} \sum_{m=0}^{9} N(m,10,n)\xi^m q^n = \sum_{m=0}^{9} \sum_{n=0}^{\infty} N(m,10,n)\xi^m q^n.$$

Expanding the last series in the above equation according to the summation index m, noting that $N(m, 10, n) = N(10-m, 10, n), \xi + \xi^3 + \xi^7 + \xi^9 = 1$ and $\xi^5 = -1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\xi q, q/\xi; q)_n} \\
&= \sum_{n\geq 0} \left(N(0,10,n) + N(1,10,n) - N(4,10,n) - N(5,10,n)\right) q^n \\
&\quad + (\xi^2 - \xi^3) \sum_{n\geq 0} \left(N(1,10,n) + N(2,10,n) - N(3,10,n) - N(4,10,n)\right) q^n. \quad (2.3.36)
\end{aligned}$$

Next, by [32, Eq.(7.6)], we know that

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, q/z; q)_n} = 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}. \quad (2.3.37)$$

Replacing the summation index n by $-n$, we find that

$$\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2+n}}{(1-zq^n)(1-z^{-1}q^n)} = \sum_{n=-\infty}^{-1} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}.$$

Substituting the above equality into (2.3.37), we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, q/z; q)_n} = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}.$$

Setting $z = \xi$ in the above equation and simplifying, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\xi q, q/\xi; q)_n} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-\xi)(1-1/\xi)(-1)^n q^{n(3n+1)/2}}{(1-\xi q^n)(1-q^n/\xi)} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1+\xi^6)(1+\xi^4)(-1)^n q^{n(3n+1)/2}}{(1+\xi^6 q^n)(1+\xi^4 q^n)} \\
&= \frac{(1+\xi^6)(1+\xi^4)}{(q;q)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)(1+\xi^2 q^n)(1+\xi^8 q^n)}{(1+\xi^6 q^n)(1+\xi^4 q^n)(1+q^n)(1+\xi^2 q^n)(1+\xi^8 q^n)} \\
&= \frac{(\xi^2+\xi^8)}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^n + q^{2n} - 1 - q^{3n})}{1+q^{5n}} \\
&\quad + \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^{3n})}{1+q^{5n}}. \qquad (2.3.38)
\end{aligned}$$

Replacing n with $-n$ in the summation index, we find that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+n}}{1+q^{5n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+3n}}{1+q^{5n}},$$

and hence the first sum on the right side of the last equality in (2.3.38) becomes

$$\frac{(\xi^2-\xi^3)}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}}.$$

Substituting this into (2.3.38), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\xi q, q/\xi; q)_n} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{5n}} + \frac{(\xi^2-\xi^3)}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}}.
\end{aligned}$$

By (2.3.40), (2.3.41) and (2.3.36), we have

$$\begin{aligned}
&\sum_{n\geq 0} \left(N(0,10,n) + N(1,10,n) - N(4,10,n) - N(5,10,n)\right) q^n \\
&+ (\xi^2-\xi^3) \sum_{n\geq 0} \left(N(1,10,n) + N(2,10,n) - N(3,10,n) - N(4,10,n)\right) q^n
\end{aligned}$$

$$= F_1(q) + (\xi^2 - \xi^3)F_2(q), \tag{2.3.39}$$

where

$$F_1(q) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{5n}} \tag{2.3.40}$$

and

$$F_2(q) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}}. \tag{2.3.41}$$

Since the coefficients of $F_1(q)$ and $F_2(q)$ are all integers and $[\mathbf{Q}(\xi) : \mathbf{Q}] = 4$, we equate the coefficient of ξ^k on both sides of (2.3.39) and find that

$$\sum_{n=0}^{\infty}(N(0,10,n) + N(1,10,n) - N(4,10,n) - N(5,10,n))q^n = F_1(q)$$

and

$$\sum_{n=0}^{\infty}(N(1,10,n) + N(2,10,n) - N(3,10,n) - N(4,10,n))q^n = F_2(q).$$

This completes the proofs of (2.1.9) and (2.1.11). Next, we prove (2.1.10) and (2.1.12). Substituting (2.3.1) and (2.3.2) into the series on the right side of (2.3.40), we find that

$$\begin{aligned} F_1(q) = &\frac{1}{(q;q)_\infty}\left\{P(q^5,-q^5) - \frac{P(q^{10},-q^5)}{q^3} + \frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}}\right\} \\ &+ \frac{1}{(q;q)_\infty}\left\{P(q^{10},-q^{10}) - q^3P(q^5,-q^{10}) - \frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+8}}{1+q^{25n+10}}\right\} \\ = &\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}} - \frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+8}}{1+q^{25n+10}} \\ &+ \frac{1}{(q;q)_\infty}\left\{P(q^5,-q^5) - \frac{P(q^{10},-q^5)}{q^3} + P(q^{10},-q^{10}) - q^3P(q^5,-q^{10})\right\}. \end{aligned} \tag{2.3.42}$$

This completes our proof of (2.1.10) after invoking Lemma 2.3.2. Similarly, substituting (2.3.1) and (2.3.3) into the series on the right side of (2.3.41), we find that

$$\begin{aligned}F_2(q) &= \frac{1}{(q;q)_\infty}\left\{\frac{P(q^5,-1)}{q^6} - \frac{P(q^{10},-1)}{q^9} - \frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+25n)/2-1}}{1+q^{25n}}\right\} \\ &\quad - \frac{1}{(q;q)_\infty}\left\{P(q^5,-q^5) - \frac{P(q^{10},-q^5)}{q^3} + \frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}}\right\} \\ &= -\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}} - \frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+25n)/2-1}}{1+q^{25n}} \\ &\quad + \frac{1}{(q;q)_\infty}\left\{\frac{P(q^5,-1)}{q^6} - \frac{P(q^{10},-1)}{q^9} - P(q^5,-q^5) + \frac{P(q^{10},-q^5)}{q^3}\right\}. \qquad (2.3.43)\end{aligned}$$

This completes our proof of (2.1.12) after invoking Lemma 2.3.3. □

2.4 Proof of Corollary 2.1.2

Proof. First, we prove (2.1.13). By (2.1.10), we find that when $i = 1$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(0,10,5n+1) + N(1,10,5n+1) - N(4,10,5n+1) - N(5,10,5n+1))q^n \\ &= \frac{J_5 J_{10}^5}{J_{1,10}J_{2,10}^2 J_{3,10}^2} = \frac{(q^5,q^{10};q^{10})_\infty}{[q;q^{10}]_\infty [q^2,q^3;q^{10}]_\infty^2} = \frac{(q^5;q^5)_\infty}{[q^2;q^5]_\infty^2 [q;q^{10}]_\infty} = \frac{L_{5,2}(q)}{[q^2;q^5]_\infty [q;q^{10}]_\infty}.\end{aligned}$$

It is clear that a product of the type $\frac{1}{1-q^m}$ has nonnegative coefficients and the factor $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients. Since, by Lemma 2.2.4, $L_{5,2}(q)$ (whose constant term is 1) has nonnegative nth coefficient for all $n \geq 1$, our inequality follows.

When $i = 2$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(0,10,5n+2) + N(1,10,5n+2) - N(4,10,5n+2) - N(5,10,5n+2))q^n \\ &= \frac{J_5 J_{10}^5}{J_{1,10}^2 J_{3,10} J_{4,10}^2} = \frac{(q^5,q^{10};q^{10})_\infty}{[q,q^4;q^{10}]_\infty^2 [q^3;q^{10}]_\infty} = \frac{(q^5;q^5)_\infty}{[q;q^5]_\infty^2 [q^3;q^{10}]_\infty} = \frac{L_{5,1}(q)}{[q;q^5]_\infty [q^3;q^{10}]_\infty}.\end{aligned}$$

Since the factor $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients, and by Lemma 2.2.4, $L_{5,1}(q)$ (whose constant term is 1) has nonnegative nth coefficient for all $n \geq 1$, our inequality follows.

When $i = 3$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(N(0,10,5n+3)+N(1,10,5n+3)-N(4,10,5n+3)-N(5,10,5n+3))q^n \\
&= \frac{J_5 J_{2,10}^2 J_{10}^5}{J_{1,10}^3 J_{4,10}^4} - \frac{1}{J_5}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+2}} \\
&= \frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}} - \frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+21n)/2+3}}{1-q^{5n+3}}, \qquad (2.4.1)
\end{aligned}$$

where the second equality follows from replacing b_1, b_2, b_3 and q by $-q^2, q^1, q^3$ and q^5, respectively, in (2.2.14). We see that to prove our inequality, it suffices to show that the first sum on the right side of the second equality in (2.4.1) has positive coefficients and the second sum on the right side of the second equality in (2.4.1) has nonnegative coefficients. We examine the first sum and find that

$$\begin{aligned}
&\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}} \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\left(\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}} - \sum_{n=0}^{\infty}\frac{q^{(15n^2+31n)/2+7}}{1-q^{5n+4}}\right) \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\left(\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}} - \frac{q^{(15n^2+31n)/2+7}}{1-q^{5n+4}}\right) \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}q^{(15n^2+9n)/2}\left(\frac{1}{1-q^{5n+1}} - \frac{q^{11n+7}}{1-q^{5n+4}}\right) \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{(1-q^{5n+1})(1-q^{5n+4})}\left\{1-q^{5n+4}-q^{11n+7}\left(1-q^{5n+1}\right)\right\} \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{(1-q^{5n+1})(1-q^{5n+4})} \\
&\quad \times\left\{\left(1-q^{5n+4}\right)\left(1-q^{11n+7}\right)+q^{16n+8}\left(1-q^3\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(q^6,q^4;q^5)_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}} \\
&\quad\times\left\{\frac{(1-q^{5n+4})\,(1-q^{11n+7})+q^{16n+8}\,(1-q^3)}{(1-q)\,(1-q^{5n+4})}\right\} \\
&= \frac{1}{(q^6,q^4;q^5)_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}}\left\{\sum_{k=0}^{11l+6}q^k+\frac{q^{16n+8}\,(1+q+q^2)}{1-q^{5n+4}}\right\}.
\end{aligned}
$$

Since the factor $\frac{1}{1-q}$ appearing in sum of the last equality has positive coefficients, we see that $\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{15n^2/2+9n/2}}{1-q^{5n+1}}$ has positive coefficients of q^n for all $n\geq 0$. Similarly, since we have

$$
\begin{aligned}
&-\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+21n)/2+3}}{1-q^{5n+3}} \\
&= \frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}} \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\left(\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}}-\frac{q^{15n^2/2+21n/2+3}}{1-q^{5n+3}}\right) \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}q^{(15n^2+19n)/2+2}\left(\frac{1}{1-q^{5n+2}}-\frac{q^{n+1}}{1-q^{5n+3}}\right) \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{(1-q^{5n+2})(1-q^{5n+3})}\left\{1-q^{5n+3}-q^{n+1}\left(1-q^{5n+2}\right)\right\} \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{(1-q^{5n+3})\,(1-q^{5n+2})} \\
&\quad\times\left\{\left(1-q^{5n+3}\right)\left(1-q^{n+1}\right)+q^{6n+3}\,(1-q)\right\} \\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}}\left\{\left(1-q^{n+1}\right)+\frac{q^{6n+3}\,(1-q)}{1-q^{5n+3}}\right\} \\
&= \frac{1}{(q^4,q^6;q^5)_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}}\left\{\sum_{k=0}^{n}q^k+\frac{q^{6n+3}}{1-q^{5n+3}}\right\},
\end{aligned}
$$

it is easy to see that $-\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{15n^2/2+21n/2+3}}{1-q^{5n+3}}$ has nonnegative coefficients. This completes the proof of the case $i=3$.

When $i = 4$, we have

$$\sum_{n=0}^{\infty}(N(0,10,5n+4)+N(1,10,5n+4)-N(4,10,5n+4)-N(5,10,5n+4))q^n$$
$$=\frac{2J_{10}^6}{J_5J_{1,10}J_{2,10}J_{3,10}J_{4,10}}=\frac{2(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty[q,q^2,q^3,q^4;q^{10}]_\infty}=\frac{2L_{10,3}(q)}{(q^5;q^{10})_\infty[q,q^2,q^4;q^{10}]_\infty}.$$

Since the factor $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients, and by Lemma 2.2.4, we know that $L_{10,3}(q)$ (whose constant term is 1) has nonnegative coefficients of q^n for all $n \geq 1$, our inequality follows. This completes the proof of (2.1.13).

Next we prove (2.1.14). By (2.1.12), we find that when $j = 1$, we have

$$\sum_{n=0}^{\infty}(N(1,10,5n+1)+N(2,10,5n+1)-N(3,10,5n+1)-N(4,10,5n+1))q^n$$
$$=\frac{2qJ_{10}^6}{J_5J_{1,10}J_{3,10}J_{4,10}^2}=\frac{2q(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty[q,q^3,q^4,q^4;q^{10}]_\infty}=\frac{2qL_{10,3}(q)}{(q^5;q^{10})_\infty[q,q^4,q^4;q^{10}]_\infty}.$$

Since the factor $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients, and by Lemma 2.2.4, we know that $L_{10,3}(q)$ whose constant term is 1 has nonnegative coefficients of q^n for all $n \geq 1$, the inequality follows.

When $j = 2$, we have

$$\sum_{n=0}^{\infty}(N(1,10,5n+2)+N(2,10,5n+2)-N(3,10,5n+2)-N(4,10,5n+2))q^n$$
$$=\frac{J_5J_{4,10}J_{10}^6}{J_{2,10}^3J_{3,10}^3}=\frac{(q^5,q^{10};q^{10})_\infty[q^4;q^{10}]_\infty}{[q^2,q^3;q^{10}]_\infty^3}=\frac{(q^5;q^5)_\infty[q^4;q^{10}]_\infty}{[q^2;q^5]_\infty^3}=\frac{L_{5,2}(q)[-q^2;q^5]_\infty}{[q^2;q^5]_\infty}.$$

Since the factor $\frac{1+q^3}{1-q^2}$ appearing on the right side of the last equality has positive coefficients of q^n for all $n \geq 2$, and by Lemma 2.2.4, we know that $L_{5,2}(q)$ (whose constant term is 1) has nonnegative coefficients of q^n for all $n \geq 1$, the inequality follows.

When $j=3$, we have

$$\sum_{n=0}^{\infty}(N(1,10,5n+3)+N(2,10,5n+3)-N(3,10,5n+3)-N(4,10,5n+3))q^n$$
$$=\frac{J_5J_{10}^5}{J_{1,10}J_{2,10}J_{3,10}^2J_{4,10}}=\frac{(q^5;q^5)_\infty}{[q,q^2,q^3,q^3,q^4;q^{10}]_\infty}=\frac{(q^5;q^5)_\infty}{[q^2;q^5]_\infty[q,q^3,q^4;q^{10}]_\infty}$$
$$=\frac{L_{5,2}(q)}{[q,q^3,q^4;q^{10}]_\infty}.$$

Since the factor $\frac{1}{1-q}$ appears on the right side of the last equality and by Lemma 2.2.4, we know that $L_{5,2}(q)$ (whose constant term is 1) has nonnegative coefficients of q^n for all $n\geq 1$, the inequality follows. Our inequality follows.

When $j=4$, we have

$$\sum_{n=0}^{\infty}(N(1,10,5n+4)+N(2,10,5n+4)-N(3,10,5n+4)-N(4,10,5n+4))q^n$$
$$=\frac{J_5J_{4,10}^2J_{5,10}J_{10}^5}{2qJ_{2,10}^4J_{3,10}^4}-\frac{1}{J_5}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(15n^2+5n)/2-1}}{1+q^{5n}}$$
$$=\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}}-\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+23n)/2+3}}{1-q^{5n+3}}$$
$$=2\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}},$$

the second equality follows from replacing b_1,b_2,b_3 and q by $-1,q^2,q^3$ and q^5, respectively, in (2.2.14) and the last equality follows from replacing n by $-n$ in the summation index in the second series on the right side of the second equality. Since we have

$$\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}}$$
$$=\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=0}^{\infty}\left(\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}}-\frac{q^{(15n^2+23n)/2+3}}{1-q^{5n+3}}\right)$$
$$=\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=0}^{\infty}q^{(15n^2+17n)/2+1}\left(\frac{1}{1-q^{5n+2}}-\frac{q^{3n+2}}{1-q^{5n+3}}\right)$$

$$
\begin{aligned}
&= \frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5} \sum_{n=0}^{\infty} \frac{q^{(15n^2+17n)/2+1}}{(1-q^{5n+2})(1-q^{5n+3})} \{1-q^{5n+3}-q^{3n+2}(1-q^{5n+2})\} \\
&= \frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5} \sum_{n=0}^{\infty} \frac{q^{(15n^2+17n)/2+1}}{(1-q^{5n+2})(1-q^{5n+3})} \left\{(1-q^{5n+3})(1-q^{3n+2})+q^{8n+4}(1-q)\right\} \\
&= \frac{[-q^2;q^5]_\infty}{(q^6,q^4;q^5)_\infty J_5} \sum_{n=0}^{\infty} \frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}} \\
&\quad \times \left\{ \frac{(1-q^{5n+3})(1-q^{3n+2})}{(1-q^{5n+3})(1-q)} + \frac{q^{8n+4}(1-q)}{(1-q^{5n+3})(1-q)} \right\} \\
&= \frac{[-q^2;q^5]_\infty}{(q^6,q^4;q^5)_\infty J_5} \sum_{n=0}^{\infty} \frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}} \left(\sum_{k=0}^{3n+1} q^k + \frac{q^{8n+4}}{1-q^{5n+3}} \right),
\end{aligned}
$$

and the factor $\frac{q(1+q)}{1-q^2}$ appearing in the sum of the last expression has positive coefficient of q^n for all $n \geq 1$, we know that $\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5} \sum_{n=-\infty}^{\infty} \frac{q^{15n^2/2+17n/2+1}}{1-q^{5n+2}}$ has positive coefficients for all $n \geq 1$, thus our inequality follows. This completes the proof of (2.1.14).

Note that, for $0 \leq t \leq 4$, we have

$$N(t,5,n) = N(t,10,n) + N(5-t,10,n). \tag{2.4.2}$$

Substituting (2.4.2) with suitable values of t into the relations between ranks modulo 5, then by comparing the results with (2.1.13) or (2.1.14), we can prove the rest of the inequalities in the corollary. For example, by (2.4.2), we have

$$N(0,5,5n+4) = N(0,10,5n+4) + N(5,10,5n+4),$$

and

$$N(1,5,5n+4) = N(1,10,5n+4) + N(4,10,5n+4).$$

Substituting these two equalities into (1.0.6) yields

$$N(0,10,5n+4) + N(5,10,5n+4) = N(1,10,5n+4) + N(4,10,5n+4). \tag{2.4.3}$$

On the other hand, by (2.1.13) (when $i = 4$), we have

$$N(0, 10, 5n + 4) + N(1, 10, 5n + 4) > N(4, 10, 5n + 4) + N(5, 10, 5n + 4). \quad (2.4.4)$$

Adding (2.4.3) to (2.4.4), we get (2.1.21). We omit the proofs of other inequalities since they are similar to the above. □

Chapter 3

M_2-ranks of partitions without repeated odd parts modulo 6 and 10

3.1 Introduction

Recall that the M_2-rank of a partition λ without repeated odd parts is defined by

$$M_2\text{-rank}(\lambda) = \left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda),$$

where $l(\lambda)$ is the largest part of λ, $n(\lambda)$ is the number of parts of λ and $\lceil . \rceil$ is the ceiling function. Let $N_2(m, n)$ denote the number of partitions of n without repeated odd parts whose M_2-rank is m. Then we have the following generating function:

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_2(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(zq^2, q^2/z; q^2)_n} \quad ([56, \text{Eq. } (1.1)]). \tag{3.1.1}$$

The two-variable generating function for M_2-rank in (3.1.1) appears many times in Ramanujan's "lost" notebook [4, Ch. 12]. As an example, the identity, [4, Entry 12.4.3],

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q^2;q^4)_nq^{2n^2}}{(-x;q^4)_{n+1}(-q^4/x;q^4)_n}+(1+1/x)\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}}$$
$$=\frac{(-xq^2;q^4)_\infty(-q^2/x;q^4)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty^2(-x;q^4)_\infty(-q^4/x;q^4)_\infty(xq;q^2)_\infty(q/x;q^2)_\infty} \tag{3.1.2}$$

gives a connection between the generating function for the M_2-rank of partitions without repeated odd parts and Ramanujan's ϕ function (the special case of $x=1$ of the infinite sum in the second series on the left side of (3.1.2)). It is also closely related to some mock theta functions. For positive integer t, we define

$$L_t(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(\xi_tq^2,q^2/\xi_t;q^2)_n},$$

where $\xi_t=e^{\frac{2\pi i}{t}}$. Then $L_2(q)$ is the second order mock theta function $\mu(-q)$ of R. J. McIntosh [61]. Also, $L_4(q)$ is the eighth order mock theta function $U_0(q)$ of B. Gordon and McIntosh [37].

Let $N_2(t,l,n)$ denote the number of partitions of n without repeated odd parts whose M_2-rank is congruent to t modulo l and for $0\le b,c,d<l$, define

$$R_{bc}(d):=\sum_{n\ge 0}(N_2(b,l,ln+d)-N_2(c,l,ln+d))q^n.$$

Then Lovejoy and Osburn [56] gave all the generating functions of $R_{bc}(d)$ for $l=3$ and 5. In this chapter, after deriving the 3-dissection of $L_6(q)$ and 5-dissection of $L_{10}(q)$, we find the following results on the generating functions for M_2-ranks modulo 6 and 10 of partition without repeated odd parts.

Theorem 3.1.1. *We have*

$$\sum_{n\ge 0}(N_2(0,6,n)+N_2(1,6,n)-N_2(2,6,n)-N_2(3,6,n))q^n$$

$$= L_6(q) \tag{3.1.3}$$

$$= \frac{J_{6,36}J_{18,36}^2J_{36}^3}{J_{3,36}J_{9,36}^3J_{15,36}} - \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{18n^2+27n+9}}{1+q^{18n+12}}$$
$$+ q\frac{J_{6,36}^2J_{18,36}J_{36}^3}{J_{3,36}^2J_{9,36}J_{15,36}^2}$$
$$+ \frac{J_{6,36}J_{18,36}^2J_{36}^3}{2qJ_{3,36}^2J_{9,36}J_{15,36}^2} - \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{18n^2+9n-1}}{1+q^{18n}}. \tag{3.1.4}$$

Theorem 3.1.2. *Let*

$$F_3(q) := \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}(1+q^{2n})}{1+q^{10n}}$$

and

$$F_4(q) := \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}(q^{4n}-1)}{1+q^{10n}}.$$

Then we have

$$\sum_{n\geq 0}(N_2(0,10,n)+N_2(1,10,n)-N_2(4,10,n)-N_2(5,10,n))q^n$$

$$= F_3(q) \tag{3.1.5}$$

$$= \frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+25n}}{1+q^{50n+10}} + 2q^5\frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3}$$
$$+ q\frac{J_{20,100}J_{30,100}^2J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^4J_{25,100}J_{35,100}^4J_{40,100}^3J_{45,100}^2}$$
$$+ q^2\frac{J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^3J_{20,100}J_{25,100}J_{35,100}^3J_{40,100}J_{45,100}^3}$$
$$+ q^3\frac{J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^4J_{15,100}^2J_{20,100}^3J_{25,100}J_{30,100}^2J_{35,100}^2J_{45,100}^4}$$
$$+ \frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+75n+24}}{1+q^{50n+30}} + 2q^4\frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^3J_{25,100}^3J_{35,100}^3J_{40,100}^2J_{45,100}^2} \tag{3.1.6}$$

and

$$\sum_{n\geq 0}(N_2(1,10,n)+N_2(2,10,n)-N_2(3,10,n)-N_2(4,10,n))q^n$$

$$=F_4(q) \tag{3.1.7}$$

$$\begin{aligned}
&=\frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^3J_{25,100}J_{35,100}^3J_{40,100}^2J_{45,100}^3}\\
&\quad-2q^5\frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3}-\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+25n}}{1+q^{50n+10}}\\
&\quad+\frac{2q^6J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{15,100}^3J_{20,100}J_{25,100}^3J_{35,100}^3J_{40,100}J_{45,100}^2}\\
&\quad+2q^7\frac{J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^3J_{25,100}^3J_{30,100}^2J_{35,100}^2J_{45,100}^3}\\
&\quad+\frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2q^3J_{25,100}J_{100}^9}-\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+25n-3}}{1+q^{50n}}\\
&\quad+\frac{q^3J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^4J_{25,100}J_{35,100}^4J_{40,100}^2J_{45,100}^2}\\
&\quad+\frac{q^4J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^3J_{20,100}^2J_{25,100}J_{30,100}J_{35,100}^3J_{45,100}^3}.
\end{aligned} \tag{3.1.8}$$

Although more and more inequalities between ranks of ordinary partitions have been studied over the years, relations between M_2-ranks of partitions without repeated odd parts have not been discussed widely. In this chapter, with the generating functions established in Theorem 3.1.1, 3.1.2 and [56, Theorem 1.1,Theorem 1.2], we discover the following inequalities. For $i=1,2,3$, $j=1,3,4$ and $n\geq 0$,

$$N_2(0,3,3n+1)\geq N_2(1,3,3n+1), \tag{3.1.9}$$

$$N_2(0,3,3n+2)\geq N_2(1,3,3n+2), \tag{3.1.10}$$

$$N_2(1,5,5n+3)\geq N_2(2,5,5n+3), \tag{3.1.11}$$

$$N_2(0,5,5n+1)\geq N_2(1,5,5n+1), \tag{3.1.12}$$

$$N_2(1,5,5n+3) \geq N_2(0,5,5n+3), \tag{3.1.13}$$

$$N_2(0,6,3n)+N_2(1,6,3n) > N_2(2,6,3n)+N_2(3,6,3n), \tag{3.1.14}$$

$$N_2(0,6,3n+1)+N_2(1,6,3n+1) > N_2(2,6,3n+1)+N_2(3,6,3n+1), \tag{3.1.15}$$

$$N_2(0,10,5n+i)+N_2(1,10,5n+i) > N_2(4,10,5n+i)+N_2(5,10,5n+i), \tag{3.1.16}$$

$$N_2(1,10,5n+j)+N_2(2,10,5n+j) \geq N_2(3,10,5n+j)+N_2(4,10,5n+j), \tag{3.1.17}$$

$$N_2(0,6,3n+1) > N_2(2,6,3n+1), \tag{3.1.18}$$

$$N_2(0,10,5n+1) > N_2(4,10,5n+1), \tag{3.1.19}$$

$$N_2(1,10,5n+1) > N_2(3,10,5n+1), \tag{3.1.20}$$

$$N_2(2,10,5n+1) > N_2(4,10,5n+1), \tag{3.1.21}$$

$$N_1(1,10,5n+3) > N_2(3,10,5n+3), \tag{3.1.22}$$

$$N_1(1,10,5n+3) > N_2(5,10,5n+3). \tag{3.1.23}$$

For (3.1.9), (3.1.10), (3.1.11), (3.1.12), (3.1.13) and (3.1.17), strict inequalities hold except in the following cases: $n=1$ in (3.1.9), $n=3$ in (3.1.10), $n=1$ and 3 in (3.1.12), $n=1,2,$ and 3 in (3.1.11) and (3.1.13), $(n,j)=(0,1)$ in (3.1.17).

Remark: From the proofs of Theorem 3.1.1 and Theorem 3.1.2, we see that

$$L_6(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+q^{2n})q^{2n^2+n}}{1+q^{6n}} \tag{3.1.24}$$

and

$$L_{10}(q) = F_3(q) + (\xi_{10}^2 - \xi_{10}^3)F_4(q).$$

Computer evidence suggests the non-negativity of coefficients of $L_6(q), F_3(q)$, and $F_4(q)$. By (3.1.14) and (3.1.15), we see that $L_6(q)$ has positive coefficients of q^n for $n \equiv 0$ and $1 \pmod 3$, $n \geq 0$. By (3.1.16), we see that $F_3(q)$ has positive coefficients of q^n for $n \equiv 1, 2,$ and $3 \pmod 5$, $n \geq 0$. And by (3.1.17), we see that $F_4(q)$ has positive coefficients of q^n for $n \equiv 1, 3,$ and $4 \pmod 5$, $n \geq 3$. We failed to prove the remaining cases and so we leave them as the following conjecture.

Conjecture 3.1.3. *For $n \geq 1$,*

$$N_2(0,6,3n+2) + N_2(1,6,3n+2) > N_2(2,6,3n+2) + N_2(3,6,3n+2),$$

$$N_2(0,10,5n) + N_2(1,10,5n) > N_2(4,10,5n) + N_2(5,10,5n),$$

$$N_2(0,10,5n+4) + N_2(1,10,5n+4) > N_2(4,10,5n+4) + N_2(5,10,5n+4),$$

$$N_2(1,10,5n) + N_2(2,10,5n) > N_2(3,10,5n) + N_2(4,10,5n),$$

$$N_2(1,10,5n+2) + N_2(2,10,5n+2) > N_2(3,10,5n+2) + N_2(4,10,5n+2).$$

This chapter is organized as follows. We give some lemmas in Section 3.2 and prove Theorems 3.1.1 and 3.1.2 in Sections 3.3 and 3.4, respectively. In Section 3.5 we prove the inequalities (3.1.9)–(3.1.23).

3.2 Some lemmas

First, we derive some theta identities used repeatedly in the proof of Theorem 3.1.2. Recalling the definition in equation (2.2.1) in Chapter 2, we have the following equivalent version of Lemma 2.2.1:

Lemma 3.2.1.

$$H(a,b,c,q)-H(a,b,d,q)=H(c,1/d,abd,q), \tag{3.2.1}$$

$$H(a,a,q/a,q^2)+H(b,b,q/b,q^2)=2H(a,q/a,b,q^2), \tag{3.2.2}$$

$$H(a,a,q/a,q^2)-H(b,b,q/b,q^2)=2H(a,q/a,b/q,q^2). \tag{3.2.3}$$

We derive the following special cases. Replacing q by q^{50} and setting $(a,b,c,d)=(q^{10},q^{25},q^{10},-q^{-5})$ in (3.2.1), we get

$$\frac{J_{15,50}^2J_{20,50}}{J_{5,50}J_{10,50}^2J_{25,50}}-\frac{q^5J_{15,50}}{J_{10,50}J_{25,50}}=\frac{\overline{J}_{10,50}\overline{J}_{15,50}J_{15,50}}{J_{5,50}\overline{J}_{5,50}J_{10,50}\overline{J}_{20,50}}.$$

Multiplying by $\frac{J_{5,50}J_{10,50}^2J_{25,50}}{J_{15,50}}$ throughout, the above equation becomes

$$J_{15,50}J_{20,50}-q^5J_{5,50}J_{10,50}=\frac{J_{5,50}J_{20,50}^2J_{25,50}}{J_{10,50}J_{15,50}}. \tag{3.2.4}$$

Replacing q by q^{50} and setting $(a,b)=(q^5,q^{15})$ in (3.2.2) and (3.2.3), respectively, we find that

$$\frac{J_{10,100}}{J_{5,100}^2J_{45,100}^2}+\frac{J_{30,100}}{J_{15,100}^2J_{35,100}^2}=\frac{2J_{20,100}J_{40,100}}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}J_{50,100}} \tag{3.2.5}$$

and

$$\frac{J_{10,100}}{J_{5,100}^2J_{45,100}^2}-\frac{J_{30,100}}{J_{15,100}^2J_{35,100}^2}=\frac{2q^5J_{10,100}J_{30,100}}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}J_{50,100}}. \tag{3.2.6}$$

Multiplying (3.2.5) by (3.2.6), we get

$$\frac{J_{10,100}^2}{J_{5,100}^4J_{45,100}^4}-\frac{J_{30,100}^2}{J_{15,100}^4J_{35,100}^4}=\frac{4q^5J_{10,100}J_{20,100}J_{30,100}J_{40,100}}{J_{5,100}^2J_{15,100}^2J_{35,100}^2J_{45,100}^2J_{50,100}^2}. \tag{3.2.7}$$

Replacing q by q^{50} and setting $(a,b)=(q^{10},q^{20})$ in (3.2.2) and (3.2.3), respectively, we find that

$$\frac{J_{20,100}}{J_{10,100}^2J_{40,100}^2}+\frac{J_{40,100}}{J_{20,100}^2J_{30,100}^2}=\frac{2}{J_{10,100}J_{20,100}J_{50,100}} \tag{3.2.8}$$

and

$$\frac{J_{20,100}}{J_{10,100}^2 J_{40,100}^2} - \frac{J_{40,100}}{J_{20,100}^2 J_{30,100}^2} = \frac{2q^{10}}{J_{30,100} J_{40,100} J_{50,100}}. \tag{3.2.9}$$

Multiplying (3.2.8) by (3.2.9), we have

$$\frac{J_{20,100}^2}{J_{10,100}^4 J_{40,100}^4} - \frac{J_{40,100}^2}{J_{20,100}^4 J_{30,100}^4} = \frac{4q^{10}}{J_{10,100} J_{20,100} J_{30,100} J_{40,100} J_{50,100}^2}. \tag{3.2.10}$$

Next, we recall [28, Theorem 1.1] which is also called Riemann Relation on page **451** of [69].

Lemma 3.2.2. *For complex parameters A, b, c, d, e satisfying $A^2 = bced$, there holds the theta function identity*

$$[A/b, A/c, A/d, A/e; q]_\infty - [b, c, d, e; q]_\infty = b[A, A/bc, A/bd, A/be; q]_\infty. \tag{3.2.11}$$

We need the following special cases. Replacing q by q^{50} and setting $(A, b, c, d, e) = (-q^{25}, q^5, q^{10}, q^{15}, q^{20})$ in (3.2.11), we have

$$\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50} - J_{5,50}J_{10,50}J_{15,50}J_{20,50} = q^5\overline{J}_{0,50}\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{25,50}. \tag{3.2.12}$$

Replacing q by q^{25} and setting $(A, b, c, d, e) = (-q^{15}, q^5, q^5, q^{10}, q^{10})$ in (3.2.11), we have

$$\overline{J}_{5,25}^2\overline{J}_{10,25}^2 - J_{5,25}^2J_{10,25}^2 = q^5\overline{J}_{0,25}^2\overline{J}_{5,25}\overline{J}_{15,25}. \tag{3.2.13}$$

Noting that

$$\frac{\overline{J}_{5,25}^2\overline{J}_{10,25}^2}{J_{25}^4} = \frac{\overline{J}_{5,50}^2\overline{J}_{10,50}^2\overline{J}_{15,50}^2\overline{J}_{20,50}^2}{J_{50}^8},$$
$$\frac{J_{5,25}^2J_{10,25}^2}{J_{25}^4} = \frac{J_{5,50}^2J_{10,50}^2J_{15,50}^2J_{20,50}^2}{J_{50}^8}$$

and

$$\frac{\overline{J}_{0,25}^2\overline{J}_{5,25}\overline{J}_{10,25}}{J_{25}^4}=\frac{\overline{J}_{0,50}^2\overline{J}_{25,50}^2\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50}}{J_{50}^8},$$

we rewrite (3.2.13) as

$$\begin{aligned}&\{\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50}-J_{5,50}J_{10,50}J_{15,50}J_{20,50}\}\times\{\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50}\\&+J_{5,50}J_{10,50}J_{15,50}J_{20,50}\}=q^5\overline{J}_{0,50}^2\overline{J}_{25,50}^2\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50}.\end{aligned}\tag{3.2.14}$$

Dividing (3.2.14) by (3.2.12), we find that

$$\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50}+J_{5,50}J_{10,50}J_{15,50}J_{20,50}=\overline{J}_{0,50}\overline{J}_{25,50}\overline{J}_{15,50}\overline{J}_{20,50}.\tag{3.2.15}$$

Next, replacing q by q^{50} and setting $(A,b,c,d,e)=(-q^{25},q^{10},q^{10},q^{15},q^{15})$ in (3.2.11), we have

$$\overline{J}_{10,50}^2\overline{J}_{15,50}^2-J_{10,50}^2J_{15,50}^2=q^{10}\overline{J}_{0,50}^2\overline{J}_{25,50}\overline{J}_{5,50}.\tag{3.2.16}$$

Replace q by q^{25} and setting $(A,b,c,d,e)=(-q^{25},q^{10},q^{10},q^{15},q^{15})$ in (3.2.11), and noting that $\overline{J}_{10,25}=\overline{J}_{15,25}$, $J_{10,25}=J_{15,25}$, we find that

$$\overline{J}_{10,25}^4-J_{10,25}^4=q^{10}\overline{J}_{0,25}^3\overline{J}_{5,25}.\tag{3.2.17}$$

Since we have

$$\frac{\overline{J}_{10,25}^4}{J_{25}^4}=\frac{\overline{J}_{10,50}^4\overline{J}_{15,50}^4}{J_{50}^8},$$
$$\frac{J_{10,25}^4}{J_{25}^4}=\frac{J_{10,50}^4J_{15,50}^4}{J_{50}^8},$$

and

$$\frac{\overline{J}_{0,25}^3\overline{J}_{5,25}}{J_{25}^4}=\frac{\overline{J}_{0,50}^3\overline{J}_{25,50}^3\overline{J}_{5,25}\overline{J}_{20,50}}{J_{50}^8}$$

we rewrite (3.2.17) as

$$\left\{\overline{J}_{10,50}^2\overline{J}_{15,50}^2 + J_{10,50}^2J_{15,50}^2\right\} \times \left\{\overline{J}_{10,50}^2\overline{J}_{15,50}^2 - J_{10,50}^2J_{15,50}^2\right\} = q^{10}\overline{J}_{0,50}^3\overline{J}_{25,50}^3\overline{J}_{5,25}\overline{J}_{20,50}. \tag{3.2.18}$$

Dividing (3.2.18) by (3.2.16), we find that

$$\overline{J}_{10,50}^2\overline{J}_{15,50}^2 + J_{10,50}^2J_{15,50}^2 = \overline{J}_{0,50}\overline{J}_{25,50}^2\overline{J}_{20,50}. \tag{3.2.19}$$

We require Lemma 3.1 in [56] for the 3-dissection and 5-dissection of $\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$:

Lemma 3.2.3.

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = (q^3,-q^6,-q^9,-q^{12},q^{15},q^{18};q^{18})_\infty - q(q^9,q^{27},q^{36};q^{36})_\infty, \tag{3.2.20}$$

and

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = (-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty$$
$$- q(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty - q^3(q^{25},q^{75},q^{100};q^{100})_\infty. \tag{3.2.21}$$

Lemma 3.2.4. *We have*

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+3n}\left[\frac{\zeta^{-4n}}{1-z^2\zeta^{-2}q^{2n}} + \frac{\zeta^{4n+6}}{1-z^2\zeta^2q^{2n}}\right]$$
$$= \frac{\zeta^2(-q,-q,\zeta^4,q^2\zeta^{-4};q^2)_\infty}{(\zeta^2,q^2\zeta^{-2},-q\zeta^{-2},-q\zeta^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{2n^2+3n}}{1-z^2q^{2n}}$$
$$+ \frac{(-z^2q,-qz^{-2},\zeta^4,q^2\zeta^{-4},\zeta^2,q^2\zeta^{-2};q^2)_\infty(q^2;q^2)_\infty^2}{(z^2\zeta^{-2},q^2\zeta^2z^{-2},\zeta^2z^2,q^2\zeta^{-2}z^{-2},-q\zeta^2,-q\zeta^{-2},z^2,q^2z^{-2};q^2)_\infty}. \tag{3.2.22}$$

The above lemma appears in [56] as Lemma 4.1. Next, we derive an identity which is similar to (3.2.22).

Lemma 3.2.5.

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}[\frac{\zeta^{-2n}}{1-z\zeta^{-1}q^{2n}}+\frac{\zeta^{2n+1}}{1-z\zeta q^{2n}}]$$
$$=\frac{[-q,\zeta^2,;q^2]_\infty}{[-q\zeta,\zeta,;q^2]_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{2n^2+n}}{1-zq^{2n}}+\frac{q[-z/q,\zeta^2,\zeta;q^2]_\infty(q^2;q^2)_\infty^2}{\zeta[z\zeta^{-1},\zeta z,-q\zeta,z;q^2]_\infty} \tag{3.2.23}$$

Proof. This follows from setting $r=1, s=3$ and replacing q, a_1, b_1, b_2 and b_3 by $q^2, -z/q, z\zeta, z/\zeta$ and z, respectively, in [21, Eq.(2.1)],

$$\frac{[a_1,..,a_r]_\infty(q;q)_\infty^2}{[b_1,..,b_s]_\infty}$$
$$=\frac{[a_1/b_1,..,a_r/b_1]_\infty(q;q)_\infty^2}{[b_2/b_1,..,b_s/b_1]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{s-r}q^{(s-r)k(k-1)/2}}{1-b_1q^k}\left(\frac{a_1..a_rb_1^{s-r-1}}{b_2..b_r}\right)^k$$
$$+\,idem(b_1;b_2,..b_s). \tag{3.2.24}$$

Here we use the usual notation

$$F(b_1,b_2,..,b_m)+idem(b_1,b_2,..,b_m)$$
$$:=F(b_1,b_2,..,b_m)+F(b_2,b_1,b_3,..,b_m)+..+F(b_m,b_2,..,b_{m-1},b_1).$$

□

Lastly, we recall the following two classical identities.

Lemma 3.2.6. *(q-binomial theorem [1, p.19, Eq.(2.2.5)]) For* $|z|<1$,

$$\sum_{n=0}^{\infty}\frac{(a;q)_nz^n}{(q;q)_n}=\frac{(az;q)_\infty}{(z;q)_\infty}.$$

Lemma 3.2.7. *(Jacobi's triple product identity [35, Eq.(2.28)])*

$$\sum_{n=-\infty}^{\infty}(-1)^na^nq^{n(n-1)/2}=(a,q/a,q;q)_\infty.$$

3.3 Proof of Theorem 3.1.1

The following lemma is used to obtain (3.3.16) which is a key identity to the proof of Theorem 3.1.1. The infinite sums on the left hand sides of equations in Lemma 3.3.1 come from $L_6(q)$ (see Eq. 3.1.24).

Lemma 3.3.1. *Let*

$$V_0 := \frac{[q^3, q^6, q^6; q^{18}]_\infty (q^{18}; q^{18})_\infty^2}{[-1, -q^6, -q^6, -q^3; q^{18}]_\infty} = \frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2 J_{18,36}^2}{2J_{36}^9}$$

and

$$V_1 := \frac{[q^9, q^6, q^6; q^{18}]_\infty (q^{18}; q^{18})_\infty^2}{q[-1, -q^6, -q^6, -q^3; q^{18}]_\infty} = \frac{J_{3,36} J_{6,36}^3 J_{9,36}^2 J_{12,36}^2 J_{15,36} J_{18,36}^2}{2qJ_{36}^9}.$$

Then we have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} = V_0 - \frac{(q^2; q^2)_\infty}{J_{9,36}(-q; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+27n+9}}{1+q^{18n+12}} \tag{3.3.1}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+3n}}{1+q^{6n}} = V_1 - \frac{(q^2; q^2)_\infty}{J_{9,36}(-q; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}. \tag{3.3.2}$$

Proof. First, we prove (3.3.1). Splitting the series on the left side of (3.3.1) into three series according to the summation index n modulo 3, we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} &= \sum_{\substack{n=-\infty\\ n\equiv 0 \pmod 3}}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} + \sum_{\substack{n=-\infty\\ n\equiv 1 \pmod 3}}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{6n}}\\ &\quad + \sum_{\substack{n=-\infty\\ n\equiv 2 \pmod 3}}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{6n}}\\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+3n}}{1+q^{18n}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+15n+3}}{1+q^{18n+6}}\\ &\quad + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+27n+10}}{1+q^{18n+12}}\\ &=: S_0 - S_1 + S_2.\end{aligned}$$

Applying (3.2.22) with q, z^2 and ζ^2 replaced by $q^9, -q^{12}$ and q^{12}, respectively, we find that

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+27n}\left(\frac{q^{-24n}}{1+q^{18n}}+\frac{q^{24n+36}}{1+q^{18n+24}}\right)\\ &= \frac{q^{12}[-q^9,q^{24};q^{18}]_\infty}{[q^{12},-q^{21};q^{18}]_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{18n^2+27n}}{1+q^{18n+12}} + \frac{[q^{21},q^6,q^{24};q^{18}]_\infty (q^{18};q^{18})_\infty^2}{[-1,-q^{24},-q^6,-q^{21};q^{18}]_\infty}.\end{aligned} \tag{3.3.3}$$

Replacing the summation index n by $n+1$, we find that

$$S_1 = -\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+51n+36}}{1+q^{18n+24}}.$$

Substituting the above equation into (3.3.3) and simplifying, we get

$$S_0 - S_1 = -\frac{[-q^9;q^{18}]_\infty}{q[-q^3;q^{18}]_\infty}S_2 + \frac{[q^3,q^6,q^{12};q^{18}]_\infty (q^{18};q^{18})_\infty^2}{[-1,-q^6,-q^6,-q^3;q^{18}]_\infty}.$$

Therefore

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{6n}}$$

$$\begin{aligned}
&= S_0 - S_1 + S_2\\
&= -\frac{[-q^9;q^{18}]_\infty}{q[-q^3;q^{18}]_\infty}S_2 + \frac{[q^3,q^6,q^{12};q^{18}]_\infty(q^{18};q^{18})_\infty^2}{[-1,-q^6,-q^6,-q^3;q^{18}]_\infty} + S_2\\
&= -\left\{(q^3,-q^6,-q^9,-q^{12},q^{15},q^{18};q^{18})_\infty - q(q^9,q^{27},q^{36};q^{36})_\infty\right\} \times \frac{(-q^9;q^{18})_\infty S_2}{q(q^{18};q^{18})_\infty}\\
&\quad + \frac{[q^3,q^6,q^{12};q^{18}]_\infty(q^{18};q^{18})_\infty^2}{[-1,-q^6,-q^6,-q^3;q^{18}]_\infty}.
\end{aligned}$$

This completes the proof of (3.3.1) after invoking (3.2.20).

Next, we prove (3.3.2). Similarly, by splitting the series on the left side of (3.3.2) into three series according to the summation index n modulo 3, we find that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+3n}}{1+q^{6n}} &= \sum_{\substack{n=-\infty\\ n\equiv 0 \pmod 3}}^{\infty}\frac{(-1)^n q^{2n^2+3n}}{1+q^{6n}} + \sum_{\substack{n=-\infty\\ n\equiv 1 \pmod 3}}^{\infty}\frac{(-1)^n q^{2n^2+3n}}{1+q^{6n}}\\
&\quad + \sum_{\substack{n=-\infty\\ n\equiv 2 \pmod 3}}^{\infty}\frac{(-1)^n q^{2n^2+3n}}{1+q^{6n}}\\
&= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+9n}}{1+q^{18n}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+21n+5}}{1+q^{18n+6}}\\
&\quad + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+33n+14}}{1+q^{18n+12}}\\
&=: S_3 - S_4 + S_5.
\end{aligned}$$

Applying (3.2.23) with q, z and ζ replaced by $q^9, -1$ and q^6, respectively, we find that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+9n}\left(\frac{q^{-12n}}{1+q^{18n-6}} + \frac{q^{12n+6}}{1+q^{18n+6}}\right)\\
&= \frac{[-q^9,q^6;q^{18}]_\infty}{[q^6,-q^3;q^{18}]_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{18n^2+9n}}{1+q^{18n}} - \frac{[q^6,q^9,q^{12};q^{18}]_\infty(q^{18};q^{18})_\infty^2}{[-1,-q^3,-q^6,-q^6;q^{18}]_\infty}. \qquad (3.3.4)
\end{aligned}$$

Replacing the summation index n by $n-1$, we have

$$S_5 = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2-3n-1}}{1+q^{18n-6}}.$$

Substituting the above equation into (3.3.4) and simplifying, we get

$$S_5 - S_4 = \frac{1}{q}\left\{\frac{[q^6, q^9, q^{12}; q^{18}]_\infty (q^{18}; q^{18})_\infty^2}{[-1, -q^3, -q^6, -q^6; q^{18}]_\infty} - \frac{[-q^9, q^6; q^{18}]_\infty}{[q^6, -q^3; q^{18}]_\infty} S_3\right\}.$$

Therefore

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+3n}}{1+q^{6n}} \\
&= S_3 - S_4 + S_5 \\
&= S_3 + \frac{1}{q}\left\{\frac{[q^6, q^9, q^{12}; q^{18}]_\infty (q^{18}; q^{18})_\infty^2}{[-1, -q^3, -q^6, -q^6; q^{18}]_\infty} - \frac{[-q^9, q^6; q^{18}]_\infty}{[q^6, -q^3; q^{18}]_\infty} S_3\right\} \\
&= -\left\{(q^3, -q^6, -q^9, -q^{12}, q^{15}, q^{18}; q^{18})_\infty - q(q^9, q^{27}, q^{36}; q^{36})_\infty\right\} \times \frac{(-q^9; q^{18})_\infty S_3}{q(q^{18}; q^{18})_\infty} \\
&\quad + \frac{[q^6, q^9, q^{12}; q^{18}]_\infty (q^{18}; q^{18})_\infty^2}{q[-1, -q^3, -q^6, -q^6; q^{18}]_\infty}.
\end{aligned}$$

This completes the proof of (3.3.2) after invoking (3.2.20). □

Next, we obtain the 3-dissection of $\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \times \{V_0 + V_1\}$ through the following lemma which gives us all the products in Theorem 3.1.1. It is used in the final step of the proof of (3.1.4).

Lemma 3.3.2.

$$V_0 + V_1 = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \times \left\{\frac{J_{6,36} J_{18,36}^2 J_{36}^3}{J_{3,36} J_{9,36}^3 J_{15,36}} + q\frac{J_{6,36}^2 J_{18,36} J_{36}^3}{J_{3,36}^2 J_{9,36} J_{15,36}^2} + \frac{J_{6,36} J_{18,36}^2 J_{36}^3}{2q J_{3,36}^2 J_{9,36} J_{15,36}^2}\right\}. \tag{3.3.5}$$

Proof. By (3.2.20), we know that

$$\frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} = \frac{J_{3,36} J_{15,36} J_{18,36}}{J_{6,36} J_{9,36}} - qJ_{9,36}.$$

Expanding the right side and comparing both sides according to the powers of q modulo 3, we find that it suffices to prove the following three identities:

$$\frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2 J_{18,36}^2}{2J_{36}^9} = \frac{J_{18,36}^3 J_{36}^3}{J_{9,36}^4} - \frac{J_{6,36} J_{18,36}^2 J_{36}^3}{2J_{3,36}^2 J_{15,36}^2}, \tag{3.3.6}$$

$$0 = q\frac{J_{6,36} J_{18,36}^2 J_{36}^3}{J_{3,36} J_{9,36}^2 J_{15,36}} - q\frac{J_{6,36} J_{18,36}^2 J_{36}^3}{J_{3,36} J_{9,36}^2 J_{15,36}}, \tag{3.3.7}$$

$$\frac{J_{3,36} J_{6,36}^3 J_{9,36}^2 J_{12,36}^2 J_{15,36} J_{18,36}^2}{2qJ_{36}^9} = \frac{J_{18,36}^3 J_{36}^3}{2qJ_{3,36} J_{9,36}^2 J_{15,36}} - q^2\frac{J_{6,36}^2 J_{18,36} J_{36}^3}{J_{3,36}^2 J_{15,36}^2}. \tag{3.3.8}$$

Simplifying each of these three identities, we see that to prove (3.3.6), it suffices to show that

$$2\frac{J_{18,36}}{J_{9,36}^4} - \frac{J_{6,36}}{J_{3,36}^2 J_{15,36}^2} = \frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2}{J_{36}^{12}}. \tag{3.3.9}$$

Noting that

$$\frac{J_{18,36} J_{36}^3}{J_{9,36}^4} = \frac{\overline{J}_{9,18}}{J_{9,18}},$$
$$\frac{J_{6,36} J_{36}^3}{J_{3,36}^2 J_{15,36}^2} = \frac{\overline{J}_{3,18}}{J_{3,18}}$$

and

$$\frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2}{J_{36}^9} = \frac{J_{3,18} J_{6,18}^2}{\overline{J}_{3,18} \overline{J}_{6,18}^2},$$

we find that (3.3.9) is equivalent to

$$\frac{2\overline{J}_{9,18}}{J_{9,18}} - \frac{\overline{J}_{3,18}}{J_{3,18}} = \frac{J_{3,18} J_{6,18}^2}{\overline{J}_{3,18} \overline{J}_{6,18}^2}.$$

Multiplying by $\frac{J_{9,18}^2}{\overline{J}_{3,18} J_{3,18} J_{6,18}}$ throughout and rearranging, the above equation becomes

$$\frac{J_{9,18}^2 J_{6,18}}{J_{3,18}^2 J_{6,18}^2} + \frac{J_{9,18}^2 J_{6,18}}{\overline{J}_{3,18}^2 \overline{J}_{6,18}^2} = \frac{2J_{9,18} \overline{J}_{6,18} \overline{J}_{9,18}}{J_{3,18} J_{6,18} \overline{J}_{3,18} \overline{J}_{6,18}}.$$

This follows from replacing q, a and b by q^9, q^3 and $-q^3$, respectively, in (3.2.2).

Equation (3.3.7) is trivial.

To prove (3.3.8), it suffices to show that

$$\frac{J_{18,36}^2}{J_{9,36}^2} - \frac{2q^3 J_{6,36}^2}{J_{3,36}J_{15,36}} = \frac{J_{3,36}^2 J_{6,36}^3 J_{9,36}^2 J_{12,36}^2 J_{15,36}^2 J_{18,36}}{J_{36}^{12}},$$

which is equivalent to

$$\frac{J_{6,12}^2}{J_{3,12}^2} - 2q\frac{J_{2,12}^2}{J_{1,12}J_{5,12}} = \frac{J_1^2 J_{2,12}}{J_{12}^3}. \tag{3.3.10}$$

Equation (3.3.10) follows from two identities:

$$\frac{J_{6,12}^2}{J_{3,12}^2} + q\frac{J_{2,12}^2}{J_{1,12}J_{5,12}} = \frac{J_{4,12}^2}{J_{1,12}J_{5,12}}, \tag{3.3.11}$$

$$\frac{J_{4,12}^2}{J_{1,12}J_{5,12}} - 3q\frac{J_{2,12}^2}{J_{1,12}J_{5,12}} = \frac{J_1^2 J_{2,12}}{J_{12}^3}. \tag{3.3.12}$$

Equation (3.3.11) follows from replacing q by q^{12} and setting

$$(A, b, c, d, e) = (q^9, q, q^5, q^6, q^6)$$

in (3.2.11).

Next, we prove (3.3.12). Multiplying throughout by $\frac{J_{1,12}J_{4,12}J_{5,12}}{J_{12}}$, equation (3.3.12) becomes

$$\frac{J_{4,12}^3}{J_{12}} - 3q\frac{J_{2,12}^2 J_{4,12}}{J_{12}} = \frac{J_1^2 J_{1,12}J_{2,12}J_{4,12}J_{5,12}}{J_{12}^4},$$

which upon simplifying and rearranging gives

$$\frac{J_1^3}{J_3} = \frac{J_4^3}{J_{12}} - 3q\frac{J_{12}^3 J_2^2}{J_4 J_6^2}.$$

Setting $b(q) = \frac{J_1^3}{J_3}$, we see that it suffices to show that

$$b(q) = b(q^4) - 3q\frac{J_{12}^3 J_2^2}{J_4 J_6^2},$$

which is proved in [40, Eq. (1.35)]. This completes the proof of the lemma. □

Now we are in a position to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. First, we prove (3.1.3). Replacing z by $\xi_6 = e^{\frac{\pi i}{3}}$ in (3.1.1), we have

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(\xi_6 q^2, q^2/\xi_6; q^2)_n} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_2(m,n)\xi_6^m q^n = \sum_{n=0}^{\infty} \sum_{m=0}^{5} \sum_{j=-\infty}^{\infty} N_2(6j+m,n)\xi_6^m q^n.$$

By the definition of $N_2(m,6,n)$, we know that

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(\xi_6 q^2, q^2/\xi_6; q^2)_n} = \sum_{m=0}^{5} \sum_{n=0}^{\infty} N_2(m,6,n)\xi_6 q^n.$$

Expanding the last series in the above equation according to the summation index m, noting that $N(m,6,n) = N(6-m,6,n), \xi_6 + \xi_6^5 = 1$ and $\xi_6^3 = -1$, we have

$$L_6(q) = \sum_{n\geq 0} (N_2(0,6,n) + N_2(1,6,n) - N_2(2,6,n) - N_2(3,6,n))q^n.$$

This completes the proof of (3.1.3).

Next, we prove (3.1.4). From a limiting case of Watson's ${}_8\phi_7$ transformation, [14, Eq. (7.2), p. 16]

$$\sum_{n=0}^{\infty} \frac{(aq/bc, d, e; q)_n (\frac{aq}{de})^n}{(q, aq/b, aq/c; q)_n}$$
$$= \frac{(aq/d, aq/e; q)_\infty}{(aq, aq/de; q)_\infty} \times \sum_{n=0}^{\infty} \frac{(a,b,c,d,e;q)_n (1-aq^{2n})(-a^2)^n q^{n(n+3)/2}}{(q, aq/b, aq/c, aq/d, aq/e; q)_n (1-a)(bcde)^n}, \qquad (3.3.13)$$

we let $e \to \infty$ and replace q, a, b, c and d by $q^2, 1, \frac{1}{z}, z$ and $-q$, respectively. Upon simplifying, we find that

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(zq^2, q^2/z; q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1-\frac{1}{z})(1-z)q^{2n^2+n}}{(1-zq^{2n})(1-q^{2n}/z)}. \qquad (3.3.14)$$

Replacing z by ξ_6 in (3.3.14), we arrive at

$$L_6(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1-\frac{1}{\xi_6})(1-\xi_6)q^{2n^2+n}}{(1-\xi_6 q^{2n})(1-q^{2n}/\xi_6)}$$

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+\xi_6^2)(1+\xi_6^4)q^{2n^2+n}}{(1+q^{2n}\xi_6^4)(1+q^{2n}\xi_6^2)}$$

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+q^{2n})q^{2n^2+n}}{1+q^{6n}}. \tag{3.3.15}$$

Substituting (3.3.1) and (3.3.2) into (3.3.15), we find that

$$L_6(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \times (V_0 + V_1) - \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+27n+9}}{1+q^{18n+12}}$$
$$- \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}. \tag{3.3.16}$$

This completes the proof after invoking Lemma 3.3.2. □

3.4 Proof of Theorem 3.1.2

First we need the following lemma on the infinite sums in $F_3(q)$ and $F_4(q)$. It is applied to prove identities (3.4.56) and (3.4.57)).

Lemma 3.4.1. *Let*

$$P_0 := \frac{q[q^{10},q^{15},q^{20};q^{50}]_\infty (q^{50};q^{50})_\infty^2}{[-q^5,-q^{10},-q^{10},-q^{20};q^{50}]_\infty} - \frac{[q^{10},q^{15},q^{20};q^{50}]_\infty (q^{50};q^{50})_\infty^2}{[-1,-q^{10},-q^{20},-q^{35};q^{50}]_\infty}$$
$$= \frac{qJ_{5,100}J_{10,100}^2J_{15,100}J_{30,100}^2J_{35,100}J_{40,100}^2J_{45,100}J_{50,100}}{J_{100}^9}$$
$$- \frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2J_{100}^9},$$

$$P_1 := \frac{[q^5,q^{10},q^{20};q^{50}]_\infty (q^{50};q^{50})_\infty^2}{[-1,-q^5,-q^{10},-q^{30};q^{50}]_\infty} - \frac{q^9[q^5,q^{10},q^{30};q^{50}]_\infty (q^{50};q^{50})_\infty^2}{[-q^{10},-q^{15},-q^{20},-q^{20};q^{50}]_\infty}$$
$$= \frac{J_{5,100}^2J_{10,100}J_{20,100}J_{30,100}^2J_{40,100}J_{45,100}^2J_{50,100}^2}{2J_{100}^9}$$
$$- \frac{q^9J_{5,100}J_{10,100}^2J_{15,100}J_{20,100}^2J_{30,100}^2J_{35,100}J_{45,100}J_{50,100}}{J_{100}^9}$$

and

$$\begin{aligned}P_2 :=& \frac{[q^{10}, q^{20}, q^{25}; q^{50}]_\infty (q^{50}; q^{50})_\infty^2}{q^3[-1, -q^{10}, -q^{10}, -q^{15}; q^{50}]_\infty} - \frac{[q^{10}, q^{20}, q^{25}; q^{50}]_\infty (q^{50}; q^{50})_\infty^2}{q^2[-q^5, -q^{20}, -q^{20}, -q^{20}; q^{50}]_\infty}\\ =& \frac{J_{10,100}^3 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^3 J_{45,100}^3 J_{50,100}^2}{2q^3 J_{20,100} J_{100}^9}\\ &- \frac{J_{5,100} J_{20,100}^3 J_{25,100}^2 J_{30,100}^3 J_{45,100} J_{50,100}^2}{2q^2 J_{40,100} J_{100}^9}.\end{aligned}$$

Then we have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{10n}} = P_0 + \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}, \tag{3.4.1}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+3n}}{1+q^{10n}} = P_1 + \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+75n+24}}{1+q^{50n+30}} \tag{3.4.2}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+5n}}{1+q^{10n}} = P_2 - \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n-3}}{1+q^{50n}}. \tag{3.4.3}$$

Proof. The proofs of the above three equations are similar to each other. We give the details of the proof of (3.4.1) and omit the rest. Splitting the series on the left of (3.4.1) into five series according to the summation index n modulo 5, we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{10n}} =& \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+5n}}{1+q^{50n}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n+3}}{1+q^{50n+10}}\\ &+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+45n+10}}{1+q^{50n+20}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+65n+21}}{1+q^{50n+30}}\\ &+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+85n+36}}{1+q^{50n+40}}\\ :=& T_0 - T_1 + T_2 - T_3 + T_4.\end{aligned}$$

We applying (3.2.23) twice. First, replacing q, z and ζ by $q^{25}, -q^{10}$ and q^{10}, respectively, we find that

$$T_0+T_2=\sum_{n=-\infty}^{\infty}(-1)^n q^{50n^2+25n}\left\{\frac{q^{-20n}}{1+q^{50n}}+\frac{q^{20n+10}}{1+q^{50n+20}}\right\}$$
$$=\frac{[q^{20},-q^{25};q^{50}]_\infty}{[q^{10},-q^{15};q^{50}]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}-\frac{[q^{10},q^{15},q^{20};q^{50}]_\infty(q^{50};q^{50})_\infty^2}{[-1,-q^{10},-q^{20},-q^{35};q^{50}]_\infty}.$$

Next, replacing q, z and ζ by $q^{25}, -q^{10}$ and q^{20}, we find that

$$\frac{1}{q}(T_3-T_4)=\sum_{n=-\infty}^{\infty}(-1)^n q^{50n^2+25n}\left\{\frac{q^{-40n}}{1+q^{50n-10}}+\frac{q^{40n+20}}{1+q^{50n+30}}\right\}$$
$$=\frac{[q^{10},-q^{25};q^{50}]_\infty}{[-q^{5},q^{20};q^{50}]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}-\frac{[q^{10},q^{15},q^{20};q^{50}]_\infty(q^{50};q^{50})_\infty^2}{[-q^{5},-q^{10},-q^{10},-q^{20};q^{50}]_\infty}.$$

Therefore

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{10n}}$$
$$=T_0-T_1+T_2-T_3+T_4$$
$$=\left\{\frac{[q^{20},-q^{25};q^{50}]_\infty}{[q^{10},-q^{15};q^{50}]_\infty}-q^3-q\frac{[q^{10},-q^{25};q^{50}]_\infty}{[-q^{5},q^{20};q^{50}]_\infty}\right\}\times\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}$$
$$+\frac{q[q^{10},q^{15},q^{20};q^{50}]_\infty(q^{50};q^{50})_\infty^2}{[-q^{5},-q^{10},-q^{10},-q^{20};q^{50}]_\infty}-\frac{[q^{10},q^{15},q^{20};q^{50}]_\infty(q^{50};q^{50})_\infty^2}{[-1,-q^{10},-q^{20},-q^{35};q^{50}]_\infty}$$
$$=\frac{1}{J_{25,100}}\left\{\frac{J_{25,100}[q^{20},-q^{25};q^{50}]_\infty}{[q^{10},-q^{15};q^{50}]_\infty}-q^3J_{25,100}-q\frac{J_{25,100}[q^{10},-q^{25};q^{50}]_\infty}{[-q^{5},q^{20};q^{50}]_\infty}\right\}$$
$$\times\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}+P_0.$$

This completes the proof of (3.4.1) after invoking (3.2.21). □

Next, we obtain the 5-dissections of $\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\times\{P_0+P_1\}$ and $\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\times\{P_2-P_0\}$ through Lemma 3.4.2 and Lemma 3.4.3, respectively. Lemma 3.4.2 (resp. Lemma3.4.3) gives us the infinite products appearing in (3.1.6) (resp.(3.1.8)). These lemmas are applied to complete the proof of Theorem 3.1.2 after we obtain (3.4.56) and (3.4.57).

Lemma 3.4.2. *Let*

$$A_0 := \frac{2q^5 J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3},$$
$$A_1 := \frac{qJ_{20,100}J_{30,100}^2J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^4J_{25,100}J_{35,100}^4J_{40,100}^3J_{45,100}^2},$$
$$A_2 := \frac{q^2J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^3J_{20,100}J_{25,100}J_{35,100}^3J_{40,100}J_{45,100}^3},$$
$$A_3 := \frac{q^3J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^4J_{15,100}^2J_{20,100}^3J_{25,100}J_{30,100}^2J_{35,100}^2J_{45,100}^4}$$

and

$$A_4 := \frac{2q^4J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^3J_{25,100}^3J_{35,100}^3J_{40,100}^2J_{45,100}^2}.$$

Then we have

$$P_0 + P_1 = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \times \{A_0 + A_1 + A_2 + A_3 + A_4\}. \tag{3.4.4}$$

Proof. By (3.2.21), we know that

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} - q\frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} - q^3J_{25,100}.$$

Expanding the right side of (3.4.4) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the following five identities.

$$\begin{aligned}
&\frac{J_{5,100}^2J_{10,100}J_{20,100}J_{30,100}^2J_{40,100}J_{45,100}^2J_{50,100}^2}{2J_{100}^9} \\
&\quad - \frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2J_{100}^9} \\
&= \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{2q^5J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3} \\
&\quad - \frac{qJ_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{2q^4J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^3J_{25,100}^3J_{35,100}^3J_{40,100}^2J_{45,100}^2}
\end{aligned}$$

$$- q^3 J_{25,100} \times \frac{q^2 J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3}, \tag{3.4.5}$$

$$\frac{q J_{5,100} J_{10,100}^2 J_{15,100} J_{30,100}^2 J_{35,100} J_{40,100}^2 J_{45,100} J_{50,100}}{J_{100}^9}$$
$$= \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{q J_{20,100} J_{30,100}^2 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^3 J_{45,100}^2}$$
$$- q \frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{2q^5 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3}$$
$$- q^3 J_{25,100} \times \frac{q^3 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100} J_{30,100}^2 J_{35,100}^2 J_{45,100}^4}, \tag{3.4.6}$$

$$0 = \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{q^2 J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3}$$
$$- \frac{q J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{q J_{20,100} J_{30,100}^2 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^3 J_{45,100}^2}$$
$$- q^3 J_{25,100} \times \frac{2q^4 J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^3 J_{25,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}^2}, \tag{3.4.7}$$

$$0 = \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{q^3 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100} J_{30,100}^2 J_{35,100}^2 J_{45,100}^4}$$
$$- \frac{q J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{q^2 J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3}$$
$$- q^3 J_{25,100} \times \frac{2q^5 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3}, \tag{3.4.8}$$

$$- \frac{q^9 J_{5,100} J_{10,100}^2 J_{15,100} J_{20,100}^2 J_{30,100}^2 J_{35,100} J_{45,100} J_{50,100}}{J_{100}^9}$$
$$= \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{2q^4 J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^3 J_{25,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}^2}$$
$$- \frac{q J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{q^3 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100} J_{30,100}^2 J_{35,100}^2 J_{45,100}^4}$$
$$- q^3 J_{25,100} \times \frac{q J_{20,100} J_{30,100}^2 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^3 J_{45,100}^2}. \tag{3.4.9}$$

Simplifying each of these five identities, we see that to prove (3.4.5), it suffices to show that

$$\frac{J_{10,100}J_{20,100}J_{30,100}J_{40,100}J_{50,100}}{2J_{100}^{24}} \left\{ J_{5,100}^2 J_{30,100} J_{45,100}^2 - J_{10,100} J_{15,100}^2 J_{35,100}^2 \right\}$$

$$= \frac{2q^5 J_{50,100}}{J_{5,100}J_{15,100}J_{20,100}J_{25,100}^4J_{35,100}J_{40,100}J_{45,100}}$$
$$\times \left\{ \frac{1}{J_{5,100}^2 J_{30,100} J_{45,100}^2} - \frac{1}{J_{10,100}J_{15,100}^2J_{35,100}^2} \right\}$$
$$- \frac{q^5}{J_{5,100}^3J_{15,100}^3J_{20,100}J_{35,100}^3J_{40,100}J_{45,100}^3}. \tag{3.4.10}$$

Multiplying by $J_{5,100}^2J_{15,100}^2J_{35,100}^2J_{45,100}^2$ throughout, (3.2.6) becomes

$$J_{5,100}^2J_{30,100}J_{45,100}^2 - J_{10,100}J_{15,100}^2J_{35,100}^2 = -\frac{2q^5J_{5,100}J_{10,100}J_{15,100}J_{30,100}J_{35,100}J_{45,100}}{J_{50,100}}. \tag{3.4.11}$$

Next, we divide by $J_{10,100}J_{30,100}$ on both sides of (3.2.6) and get

$$\frac{1}{J_{5,100}^2J_{30,100}J_{45,100}^2} - \frac{1}{J_{10,100}J_{15,100}^2J_{35,100}^2} = \frac{2q^5}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}J_{50,100}}. \tag{3.4.12}$$

Substituting (3.4.11) and (3.4.12) into (3.4.10), we find that (3.4.10) is equivalent to

$$- q^5\frac{J_{5,100}J_{10,100}^2J_{15,100}J_{20,100}J_{30,100}^2J_{35,100}J_{40,100}J_{45,100}}{J_{100}^{24}}$$
$$= \frac{q^5}{J_{5,100}^2J_{15,100}^2J_{20,100}J_{35,100}^2J_{40,100}J_{45,100}^2} \times \left\{ \frac{4q^5}{J_{25,100}^4} - \frac{1}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}} \right\}. \tag{3.4.13}$$

Multiplying by $\frac{1}{q^5}J_{5,100}J_{15,100}J_{20,100}J_{35,100}J_{40,100}J_{45,100}J_{100}^8$ throughout and noting that

$$-\frac{J_{5,100}^2J_{10,100}^2J_{15,100}^2J_{20,100}^2J_{30,100}^2J_{35,100}^2J_{40,100}^2J_{45,100}^2}{J_{100}^{16}} = -\frac{J_{5,25}^2J_{10,25}^2}{J_{25}^4},$$
$$\frac{4q^5J_{100}^8}{J_{5,100}J_{15,100}J_{25,100}^4J_{35,100}J_{45,100}} = \frac{q^5\overline{J}_{0,25}^2\overline{J}_{5,25}\overline{J}_{10,25}}{J_{25}^4}$$

and

$$-\frac{J_{100}^8}{J_{5,100}^2J_{15,100}^2J_{35,100}^2J_{45,100}^2} = -\frac{\overline{J}_{5,25}^2\overline{J}_{10,25}^2}{J_{25}^4},$$

we find that (3.4.13) is equivalent to

$$-J_{5,25}^2 J_{10,25}^2 = q^5 \overline{J}_{0,25}^2 \overline{J}_{5,25} \overline{J}_{10,25} - \overline{J}_{5,25}^2 \overline{J}_{10,25}^2.$$

which follows from (3.2.12). This completes the proof of (3.4.5).

To prove (3.4.6), it suffices to show that

$$\begin{aligned}
&\frac{J_{5,100} J_{10,100}^2 J_{15,100} J_{30,100}^2 J_{35,100} J_{40,100}^2 J_{45,100} J_{50,100}}{J_{100}^9} \\
&= \frac{J_{20,100}^2 J_{30,100}^2 J_{50,100}^2 J_{100}^{15}}{J_{5,100}^2 J_{10,100}^3 J_{15,100}^3 J_{25,100}^2 J_{35,100}^3 J_{40,100}^4 J_{45,100}^2} \\
&\quad - 2q^5 \frac{J_{10,100} J_{40,100} J_{50,100}^2 J_{100}^{15}}{J_{5,100}^2 J_{15,100}^2 J_{20,100}^3 J_{25,100}^4 J_{30,100}^2 J_{35,100}^2 J_{45,100}^2} \\
&\quad - q^5 \frac{J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{30,100}^2 J_{35,100}^2 J_{45,100}^4}.
\end{aligned}$$

Multiplying by $-\frac{J_{5,100}^2 J_{15,100}^2 J_{20,100}^3 J_{30,100}^2 J_{35,100}^2 J_{45,100}^2}{J_{10,100} J_{40,100} J_{50,100} J_{100}^{15}}$ throughout and rearranging, the above equation becomes

$$\begin{aligned}
&2q^5 \frac{J_{50,100}}{J_{25,100}^4} + q^5 \frac{J_{10,100}}{J_{5,100}^2 J_{45,100}^2} - \frac{J_{20,100}^5 J_{30,100}^4 J_{50,100}}{J_{10,100}^4 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^5} \\
&= -\frac{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{20,100}^3 J_{30,100}^4 J_{35,100}^3 J_{40,100} J_{45,100}^3}{J_{100}^{24}}.
\end{aligned} \tag{3.4.14}$$

Next, we multiply by $\frac{J_{20,100}^3 J_{30,100}^4 J_{50,100}}{J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}}$ on both sides of (3.2.10) and get

$$\begin{aligned}
&\frac{J_{20,100}^5 J_{30,100}^4 J_{50,100}}{J_{10,100}^4 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^5} \\
&= \frac{J_{40,100} J_{50,100}}{J_{15,100} J_{20,100} J_{25,100}^2 J_{35,100}} + \frac{4q^{10} J_{20,100}^2 J_{30,100}^3}{J_{10,100} J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^2 J_{50,100}}.
\end{aligned} \tag{3.4.15}$$

Substituting (3.4.15) into (3.4.14), we find that (3.4.14) is equivalent to

$$\begin{aligned}
&\left\{2q^5 \frac{J_{50,100}}{J_{25,100}^4} - \frac{J_{40,100} J_{50,100}}{J_{15,100} J_{20,100} J_{25,100}^2 J_{35,100}}\right\} \\
&+ \left\{q^5 \frac{J_{10,100}}{J_{5,100}^2 J_{45,100}^2} - \frac{4q^{10} J_{20,100}^2 J_{30,100}^3}{J_{10,100} J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^2 J_{50,100}}\right\}
\end{aligned}$$

$$= -\frac{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{20,100}^3 J_{30,100}^4 J_{35,100}^3 J_{40,100} J_{45,100}^3}{J_{100}^{24}}. \tag{3.4.16}$$

The above identity follows from the following three identities:

$$2q^5 \frac{J_{50,100}}{J_{25,100}^4} - \frac{J_{40,100} J_{50,100}}{J_{15,100} J_{20,100} J_{25,100}^2 J_{35,100}}$$
$$= -\frac{J_{5,100}^2 J_{10,100} J_{15,100} J_{30,100} J_{35,100} J_{40,100}^2 J_{45,100}^2 J_{50,100}}{J_{25,100}^2 J_{100}^{12}} \tag{3.4.17}$$

$$q^5 \frac{J_{10,100}}{J_{5,100}^2 J_{45,100}^2} - \frac{4q^{10} J_{20,100}^2 J_{30,100}^3}{J_{10,100} J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^2 J_{50,100}} = \frac{q^5 J_{5,100}^2 J_{20,100}^4 J_{30,100}^4 J_{45,100}^2}{J_{10,100} J_{40,100}^2 J_{100}^{12}}, \tag{3.4.18}$$

$$\frac{q^5 J_{5,100}^2 J_{20,100}^4 J_{30,100}^4 J_{45,100}^2}{J_{10,100} J_{40,100}^2 J_{100}^{12}} - \frac{J_{5,100}^2 J_{10,100} J_{15,100} J_{30,100} J_{35,100} J_{40,100}^2 J_{45,100}^2 J_{50,100}}{J_{25,100}^2 J_{100}^{12}}$$
$$= -\frac{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{20,100}^3 J_{30,100}^4 J_{35,100}^3 J_{40,100} J_{45,100}^3}{J_{100}^{24}}. \tag{3.4.19}$$

By substituting (3.4.17) and (3.4.18) into (3.4.19), we can recover (3.4.16). Hence it suffices to prove (3.4.17), (3.4.18) and (3.4.19).

First, we prove (3.4.17). Multiplying by $\frac{J_{25,100}^2 J_{100}^2}{J_{50,100}}$ throughout and noting that

$$\frac{2q^5 J_{100}^2}{J_{25,100}^2} = \frac{q^5 \overline{J}_{0,50} \overline{J}_{25,50}}{J_{50}^2},$$
$$-\frac{J_{40,100} J_{100}^2}{J_{15,100} J_{20,100} J_{35,100}} = -\frac{\overline{J}_{20,50} \overline{J}_{15,50}}{J_{50}^2}$$

and

$$-\frac{J_{5,100}^2 J_{10,100} J_{15,100} J_{30,100} J_{35,100} J_{40,100}^2 J_{45,100}^2}{J_{100}^{10}} = -\frac{J_{5,50} J_{10,50} J_{15,50} J_{20,50}}{\overline{J}_{5,50} \overline{J}_{10,50} J_{50}^2},$$

we find that (3.4.17) is equivalent to

$$q^5 \overline{J}_{0,50} \overline{J}_{25,50} - \overline{J}_{15,50} \overline{J}_{20,50} = -\frac{J_{5,50} J_{10,50} J_{15,50} J_{20,50}}{\overline{J}_{5,50} \overline{J}_{10,50}}.$$

This follows from dividing by $\overline{J}_{5,50}\overline{J}_{10,50}$ on both sides of (3.2.12).

Next, we prove (3.4.18). Multiplying by $\frac{J_{10,100}J_{100}^6}{q^5 J_{5,100}^2 J_{45,100}^2}$ throughout and noting that

$$\frac{J_{10,100}^2 J_{100}^6}{J_{5,100}^4 J_{45,100}^4} = \frac{\overline{J}_{5,50}^2}{J_{5,50}^2},$$

$$-\frac{4q^5 J_{20,100}^2 J_{30,100}^3 J_{100}^6}{J_{5,100}^2 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^2 J_{45,100}^2 J_{50,100}} = -\frac{q^5 \overline{J}_{25,50}\overline{J}_{15,50}\overline{J}_{0,50}^2}{J_{5,50}^2 \overline{J}_{20,50}^2}$$

and

$$\frac{J_{20,100}^4 J_{30,100}^4}{J_{40,100}^2 J_{100}^6} = \frac{J_{20,50}^2}{\overline{J}_{20,50}^2},$$

we find that (3.4.18) is equivalent to

$$\frac{\overline{J}_{5,50}^2}{J_{5,50}^2} - \frac{q^5 \overline{J}_{25,50}\overline{J}_{15,50}\overline{J}_{0,50}^2}{J_{5,50}^2 \overline{J}_{20,50}^2} = \frac{J_{20,50}^2}{\overline{J}_{20,50}^2}.$$

Multiplying by $J_{5,50}^2\overline{J}_{20,50}^2$ throughout and rearranging, the above equation gives

$$\overline{J}_{5,50}^2\overline{J}_{20,50}^2 - J_{5,50}^2 J_{20,50}^2 = q^5 \overline{J}_{25,50}\overline{J}_{15,50}\overline{J}_{0,50}^2.$$

This follows from replacing q by q^{50} and setting $(A, b, c, d, e) = (-q^{25}, q^5, q^5, q^{20}, q^{20})$ in (3.2.11).

Finally, we prove (3.4.19). Multiplying by $\frac{J_{100}^{16}}{J_{5,100}^3 J_{15,100} J_{20,100}^2 J_{30,100}^3 J_{35,100} J_{45,100}^3}$ throughout and noting that

$$\frac{q^5 J_{20,100}^2 J_{30,100} J_{100}^4}{J_{5,100}J_{10,100}J_{15,100}J_{35,100}J_{40,100}^2 J_{45,100}} = \frac{q^5 \overline{J}_{5,50}\overline{J}_{10,50}^2\overline{J}_{15,50}}{J_{50}^4},$$

$$-\frac{J_{10,100}J_{40,100}^2 J_{50,100} J_{100}^4}{J_{5,100}J_{20,100}^2 J_{25,100}^2 J_{30,100}^2 J_{45,100}} = -\frac{\overline{J}_{5,50}\overline{J}_{20,50}^2\overline{J}_{25,50}}{J_{50}^4}$$

and

$$-\frac{J_{10,100}J_{15,100}^2 J_{20,100}J_{30,100}J_{35,100}^2 J_{40,100}}{J_{100}^8} = -\frac{J_{10,50}J_{15,50}^2 J_{20,50}}{J_{50}^4},$$

we find that (3.4.19) is equivalent to

$$q^5\overline{J}_{5,50}\overline{J}_{10,50}^2\overline{J}_{15,50} - \overline{J}_{5,50}\overline{J}_{20,50}^2\overline{J}_{25,50} = -J_{10,50}J_{15,50}^2J_{20,50},$$

which follows from replacing q by q^{50} and setting

$$(A, b, c, d, e) = (-q^{35}, -q^5, -q^{20}, -q^{20}, -q^{25})$$

in (3.2.11). This completes the proof of (3.4.6).

To prove (3.4.7), it suffices to show that

$$\begin{aligned}0 = &\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^2J_{25,100}^2J_{35,100}^2J_{40,100}^2J_{45,100}^3}\\ &- \frac{J_{30,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}J_{10,100}^2J_{15,100}^4J_{25,100}^2J_{35,100}^4J_{40,100}^2J_{45,100}}\\ &- 2q^5\frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^3J_{25,100}^2J_{35,100}^3J_{40,100}^2J_{45,100}^2},\end{aligned}$$

which follows from multiplying by $\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}J_{10,100}^2J_{15,100}^2J_{25,100}^2J_{35,100}^2J_{40,100}^2J_{45,100}}$ on both sides of (3.2.6).

To prove (3.4.8), it suffices to show that

$$\begin{aligned}0 = &\frac{J_{10,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^4J_{15,100}J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{35,100}J_{45,100}^4}\\ &- \frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}^3J_{20,100}^2J_{25,100}^2J_{30,100}J_{35,100}^3J_{45,100}^2}\\ &- 2q^5\frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^2J_{30,100}J_{35,100}^2J_{40,100}^2J_{45,100}^3},\end{aligned}$$

which follows from multiplying by $\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{35,100}J_{40,100}^2J_{45,100}^2}$ on both sides of (3.2.6).

To prove (3.4.9), it suffices to show that

$$-\frac{q^5J_{5,100}J_{10,100}^2J_{15,100}J_{20,100}^2J_{30,100}^2J_{35,100}J_{45,100}J_{50,100}}{J_{100}^9}$$

$$= \frac{2J_{20,100}J_{30,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^2J_{25,100}^4J_{35,100}^2J_{40,100}^3J_{45,100}^2} - \frac{J_{10,100}^2J_{40,100}^2J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^4J_{25,100}^2J_{30,100}^3J_{35,100}^2J_{45,100}^3} - \frac{J_{20,100}J_{30,100}^2J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^4J_{35,100}^4J_{40,100}^3J_{45,100}^2}.$$

Multiplying by $-\frac{J_{5,100}^2J_{10,100}^2J_{15,100}^2J_{35,100}^2J_{40,100}^3J_{45,100}^2}{J_{20,100}J_{30,100}J_{50,100}J_{100}^{15}}$ throughout and rearranging, the above equation becomes

$$q^5\frac{J_{5,100}^3J_{10,100}^4J_{15,100}^3J_{20,100}J_{30,100}J_{35,100}^3J_{40,100}^3J_{45,100}^3}{J_{100}^{24}} = \frac{J_{10,100}^4J_{40,100}^5J_{50,100}}{J_{5,100}J_{20,100}^5J_{25,100}^2J_{30,100}^4J_{45,100}} - 2\frac{J_{50,100}}{J_{25,100}^4} + \frac{J_{30,100}}{J_{15,100}^2J_{35,100}^2}. \tag{3.4.20}$$

Next, we multiply by $\frac{J_{10,100}^4J_{40,100}^3J_{50,100}}{J_{5,100}J_{20,100}J_{25,100}^2J_{45,100}}$ on both sides of (3.2.10) and get

$$\frac{J_{10,100}^4J_{40,100}^5J_{50,100}}{J_{5,100}J_{20,100}^5J_{25,100}^2J_{30,100}^4J_{45,100}} = \frac{J_{20,100}J_{50,100}}{J_{5,100}J_{25,100}^2J_{40,100}J_{45,100}} - \frac{4q^{10}J_{10,100}^3J_{40,100}^2}{J_{5,100}J_{20,100}^2J_{25,100}^2J_{30,100}J_{45,100}J_{50,100}}. \tag{3.4.21}$$

Substituting (3.4.21) into (3.4.20), we find that (3.4.20) is equivalent to

$$\left\{\frac{J_{20,100}J_{50,100}}{J_{5,100}J_{25,100}^2J_{40,100}J_{45,100}} - 2\frac{J_{50,100}}{J_{25,100}^4}\right\} + \left\{\frac{J_{30,100}}{J_{15,100}^2J_{35,100}^2} - \frac{4q^{10}J_{10,100}^3J_{40,100}^2}{J_{5,100}J_{20,100}^2J_{25,100}^2J_{30,100}J_{45,100}J_{50,100}}\right\} = q^5\frac{J_{5,100}^3J_{10,100}^4J_{15,100}^3J_{20,100}J_{30,100}J_{35,100}^3J_{40,100}^3J_{45,100}^3}{J_{100}^{24}}. \tag{3.4.22}$$

This follows from the following three identities:

$$\frac{J_{20,100}J_{50,100}}{J_{5,100}J_{25,100}^2J_{40,100}J_{45,100}} - 2\frac{J_{50,100}}{J_{25,100}^4} = -\frac{J_{5,100}J_{10,100}J_{15,100}^2J_{20,100}^2J_{30,100}J_{35,100}^2J_{45,100}J_{50,100}}{J_{25,100}^2J_{100}^{12}}, \tag{3.4.23}$$

$$\frac{J_{30,100}}{J_{15,100}^2 J_{35,100}^2} - \frac{4q^{10} J_{10,100}^3 J_{40,100}^2}{J_{5,100} J_{20,100}^2 J_{25,100}^2 J_{30,100} J_{45,100} J_{50,100}} = \frac{J_{10,100}^4 J_{15,100}^2 J_{35,100}^2 J_{40,100}^4}{J_{20,100}^2 J_{30,100} J_{100}^{12}}, \tag{3.4.24}$$

$$- \frac{J_{5,100} J_{10,100} J_{15,100}^2 J_{20,100}^2 J_{30,100} J_{35,100}^2 J_{45,100} J_{50,100}}{J_{25,100}^2 J_{100}^{12}} + \frac{J_{10,100}^4 J_{15,100}^2 J_{35,100}^2 J_{40,100}^4}{J_{20,100}^2 J_{30,100} J_{100}^{12}}$$
$$= q^5 \frac{J_{5,100}^3 J_{10,100}^4 J_{15,100}^3 J_{20,100} J_{30,100} J_{35,100}^3 J_{40,100}^3 J_{45,100}^3}{J_{100}^{24}}. \tag{3.4.25}$$

By substituting (3.4.23) and (3.4.24) into (3.4.25), we can recover (3.4.22). Hence it suffices to prove (3.4.23), (3.4.24) and (3.4.25).

First, we prove (3.4.23). Multiplying by $\frac{J_{25,100}^2 J_{100}^2}{J_{50,100}}$ throughout and noting that

$$\frac{J_{20,100} J_{100}^2}{J_{5,100} J_{40,100} J_{45,100}} = \frac{\overline{J}_{5,50} \overline{J}_{10,50}}{J_{50}^2},$$
$$-\frac{2J_{100}^2}{J_{25,100}^2} = -\frac{\overline{J}_{0,50} \overline{J}_{25,50}}{J_{50}^2}$$

and

$$-\frac{J_{5,100} J_{10,100} J_{15,100}^2 J_{20,100}^2 J_{30,100} J_{35,100}^2 J_{45,100}}{J_{100}^{10}} = -\frac{J_{5,50} J_{10,50} J_{15,50} J_{20,50}}{\overline{J}_{15,50} \overline{J}_{20,50} J_{50}^2},$$

we find that (3.4.23) is equivalent to

$$\overline{J}_{5,50} \overline{J}_{10,50} - \overline{J}_{0,50} \overline{J}_{25,50} = -\frac{J_{5,50} J_{10,50} J_{15,50} J_{20,50}}{\overline{J}_{15,50} \overline{J}_{20,50}}. \tag{3.4.26}$$

This follows from dividing by $\overline{J}_{15,50} \overline{J}_{20,50}$ on both sides of (3.2.15).

Next, we prove (3.4.24). Multiplying by $\frac{J_{20,100}^2 J_{30,100} J_{100}^4}{J_{10,100}^2 J_{40,100}^2}$ throughout and noting that

$$\frac{J_{20,100}^2 J_{30,100}^2 J_{100}^4}{J_{10,100}^2 J_{15,100}^2 J_{35,100}^2 J_{40,100}^2} = \frac{\overline{J}_{10,50}^2 \overline{J}_{15,50}^2}{J_{50}^4},$$
$$-\frac{4q^{10} J_{10,100} J_{100}^4}{J_{5,100} J_{25,100}^2 J_{45,100} J_{50,100}} = -\frac{q^{10} \overline{J}_{5,50} \overline{J}_{25,50} \overline{J}_{0,50}^2}{J_{50}^4}$$

and

$$\frac{J_{10,100}^2 J_{15,100}^2 J_{35,100}^2 J_{40,100}^2}{J_{100}^8} = \frac{J_{10,50}^2 J_{15,50}^2}{J_{50}^4},$$

we find that (3.4.24) is equivalent to

$$\overline{J}_{10,50}^2 \overline{J}_{15,50}^2 - q^{10} \overline{J}_{5,50} \overline{J}_{25,50} \overline{J}_{0,50}^2 = J_{10,50}^2 J_{15,50}^2.$$

This follows from (3.2.16).

Finally, we prove (3.4.25). Multiplying by $-\frac{J_{100}^{16}}{J_{5,100}J_{10,100}^3 J_{15,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}}$ throughout and noting that

$$\frac{J_{20,100}^2 J_{30,100} J_{50,100} J_{100}^4}{J_{10,100}^2 J_{15,100} J_{35,100} J_{25,100}^2 J_{40,100}^2} = \frac{\overline{J}_{10,50}^2 \overline{J}_{15,50} \overline{J}_{25,50}}{J_{50}^4},$$

$$-\frac{J_{10,100} J_{40,100}^2 J_{100}^4}{J_{5,100} J_{15,100} J_{20,100}^2 J_{30,100} J_{35,100} J_{45,100}} = -\frac{\overline{J}_{5,50} \overline{J}_{15,50} \overline{J}_{20,50}^2}{J_{50}^4}$$

and

$$-\frac{q^5 J_{5,100}^2 J_{10,100} J_{20,100} J_{30,100} J_{40,100} J_{45,100}^2}{J_{100}^8} = -\frac{q^5 J_{5,50}^2 J_{10,50} J_{20,50}}{J_{50}^4},$$

we find that (3.4.25) is equivalent to

$$\overline{J}_{10,50}^2 \overline{J}_{15,50} \overline{J}_{25,50} - \overline{J}_{5,50} \overline{J}_{15,50} \overline{J}_{20,50}^2 = -q^5 J_{5,50}^2 J_{10,50} J_{20,50},$$

which follows from replacing q by q^{50} and setting

$$(A, b, c, d, e) = (q^{30}, -q^5, -q^{20}, -q^{20}, -q^{15})$$

in (3.2.11). This completes the proof of the lemma. □

Lemma 3.4.3. *Let*

$$B_0 := \frac{J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{25,100} J_{35,100}^3 J_{40,100}^2 J_{45,100}^3}$$

$$- 2q^5 \frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3},$$

$$B_1 := \frac{2q^6 J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{15,100}^3J_{20,100}J_{25,100}^3J_{35,100}^3J_{40,100}J_{45,100}^2},$$

$$B_2 := 2q^7 \frac{J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^3J_{25,100}^3J_{30,100}^2J_{35,100}^2J_{45,100}^3} + \frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2q^3J_{25,100}J_{100}^9},$$

$$B_3 := \frac{q^3 J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^4J_{25,100}J_{35,100}^4J_{40,100}^2J_{45,100}^2}$$

and

$$B_4 := \frac{q^4 J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^3J_{20,100}^2J_{25,100}J_{30,100}J_{35,100}^3J_{45,100}^3}.$$

Then we have

$$P_2 - P_0 = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \times \{B_0 + B_1 + B_2 + B_3 + B_4\}. \tag{3.4.27}$$

Proof. Expanding the right side of (3.4.27) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the five identities:

$$\begin{aligned}
&\frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2J_{100}^9} \\
&= \left\{ \frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^3J_{25,100}J_{35,100}^3J_{40,100}^2J_{45,100}^3} \right. \\
&\quad \left. - 2q^5 \frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3} \right\} \times \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \\
&\quad - q^3 J_{25,100} \times \left\{ \frac{2q^7 J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^3J_{25,100}^3J_{30,100}^2J_{35,100}^2J_{45,100}^3} \right. \\
&\quad \left. + \frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2q^3J_{25,100}J_{100}^9} \right\} \\
&\quad - \frac{qJ_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{q^4 J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^3J_{20,100}^2J_{25,100}J_{30,100}J_{35,100}^3J_{45,100}^3},
\end{aligned} \tag{3.4.28}$$

$$
\begin{aligned}
&- \frac{qJ_{5,100}J_{10,100}^2J_{15,100}J_{30,100}^2J_{35,100}J_{40,100}^2J_{45,100}J_{50,100}}{J_{100}^9} \\
=&\frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{2q^6J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{15,100}^3J_{20,100}J_{25,100}^3J_{35,100}^3J_{40,100}J_{45,100}^2} \\
&- \left\{ \frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^3J_{25,100}J_{35,100}^3J_{40,100}^2J_{45,100}^3} \right. \\
&\quad \left. - \frac{2q^5J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3} \right\} \times \frac{qJ_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \\
&\quad - q^3J_{25,100} \times \frac{q^3J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^4J_{25,100}J_{35,100}^4J_{40,100}^2J_{45,100}^2}, \qquad (3.4.29)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{J_{10,100}^3J_{15,100}J_{25,100}^2J_{35,100}J_{40,100}^3J_{45,100}^3J_{50,100}^2}{2q^3J_{20,100}J_{100}^9} \\
&= \left\{ \frac{2q^7J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^3J_{25,100}^3J_{30,100}^2J_{35,100}^2J_{45,100}^3} \right. \\
&\quad \left. + \frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2q^3J_{25,100}J_{100}^9} \right\} \times \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \\
&\quad - q^3J_{25,100} \times \frac{q^4J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^3J_{20,100}^2J_{25,100}J_{30,100}J_{35,100}^3J_{45,100}^3} \\
&\quad - \frac{qJ_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{2q^6J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{15,100}^3J_{20,100}J_{25,100}^3J_{35,100}^3J_{40,100}J_{45,100}^2},
\end{aligned}
$$

(3.4.30)

$$
\begin{aligned}
&- \frac{J_{5,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{45,100}J_{50,100}^2}{2q^2J_{40,100}J_{100}^9} \\
&= \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{q^3J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^4J_{25,100}J_{35,100}^4J_{40,100}^2J_{45,100}^2} \\
&\quad - \left\{ \frac{2q^7J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^3J_{25,100}^3J_{30,100}^2J_{35,100}^2J_{45,100}^3} \right. \\
&\quad \left. + \frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{2q^3J_{25,100}J_{100}^9} \right\} \times q\frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \\
&\quad - q^3J_{25,100} \times \left\{ \frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^3J_{25,100}J_{35,100}^3J_{40,100}^2J_{45,100}^3} \right. \\
&\quad \left. - \frac{2q^5J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3} \right\}, \qquad (3.4.31)
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{q^4 J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{25,100}J_{30,100}J_{35,100}^3 J_{45,100}^3} \\
& - q\frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{q^3 J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2 J_{10,100}J_{15,100}^4 J_{25,100}J_{35,100}^4 J_{40,100}^2 J_{45,100}^2} \\
& - q^3 J_{25,100} \times \frac{2q^6 J_{50,100}J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100}J_{25,100}^3 J_{35,100}^3 J_{40,100}J_{45,100}^2}. \qquad (3.4.32)
\end{aligned}
$$

Simplifying each of these five identities, we see that to prove (3.4.28), it suffices to show that

$$
\begin{aligned}
& \frac{J_{10,100}^2 J_{15,100}^2 J_{20,100}J_{30,100}J_{35,100}^2 J_{40,100}J_{50,100}^2}{J_{100}^9} \\
& = \left\{ \frac{J_{20,100}J_{30,100}J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{10,100}^2 J_{15,100}^2 J_{25,100}^2 J_{35,100}^2 J_{40,100}^3 J_{45,100}^3} \right. \\
& \qquad \left. - \frac{2q^5 J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{15,100}J_{20,100}J_{25,100}^4 J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3} \right\} \\
& \qquad - \left\{ \frac{q^5 J_{10,100}J_{40,100}J_{50,100}^2 J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100}^3 J_{25,100}^2 J_{30,100}^2 J_{35,100}^3 J_{45,100}^2} \right. \\
& \qquad \left. + \frac{2q^{10} J_{10,100}^2 J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^3 J_{25,100}^2 J_{30,100}^2 J_{35,100}^2 J_{45,100}^3} \right\}. \qquad (3.4.33)
\end{aligned}
$$

Replacing q by $q^{1/2}$ in (3.2.9), we have

$$
\frac{J_{10,50}}{J_{5,50}^2 J_{20,50}^2} - \frac{J_{20,50}}{J_{10,50}^2 J_{15,50}^2} = \frac{2q^5}{J_{15,50}J_{20,50}J_{25,50}}.
$$

Noting that $\frac{J_{k,50}}{J_{50}} = \frac{J_{k,100}J_{50-k,100}}{J_{100}^2}$ and rearranging, the above equation gives

$$
\frac{J_{10,100}J_{40,100}}{J_{5,100}^2 J_{20,100}^2 J_{30,100}^2 J_{45,100}^2} = \frac{J_{20,100}J_{30,100}}{J_{10,100}^2 J_{15,100}^2 J_{35,100}^2 J_{40,100}^2} + \frac{2q^5}{J_{15,100}J_{20,100}J_{25,100}^2 J_{30,100}J_{35,100}}. \qquad (3.4.34)
$$

Multiplying by $\frac{J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{25,100}^2 J_{40,100}J_{45,100}^3}$ throughout and subtracting

$$
\frac{4q^5 J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{15,100}J_{20,100}J_{25,100}^4 J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}
$$

from both sides, (3.4.34) becomes

$$\frac{J_{20,100}J_{30,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{10,100}^2J_{15,100}^2J_{25,100}^2J_{35,100}^2J_{40,100}^3J_{45,100}^3} - \frac{2q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}$$
$$= \frac{J_{10,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^5J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{45,100}^5} - \frac{4q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}. \tag{3.4.35}$$

We note that the left side of the above equation is the sum of the first two terms on the right side of (3.4.33). Next, multiplying by $\frac{q^5J_{10,100}J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^2}$ on both sides of (3.2.6) and rearranging, we get

$$\frac{q^5J_{10,100}J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}^3J_{20,100}^3J_{25,100}^2J_{30,100}^2J_{35,100}^3J_{45,100}^2} + \frac{2q^{10}J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^3J_{25,100}^2J_{30,100}^2J_{35,100}^2J_{45,100}^3}$$
$$= \frac{q^5J_{10,100}^2J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^4J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^4}. \tag{3.4.36}$$

Substituting (3.4.35) and (3.4.36) into (3.4.33), we find that (3.4.33) is equivalent to

$$\frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{J_{100}^9}$$
$$= \frac{J_{10,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^5J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{45,100}^5} - \frac{4q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}$$
$$- \frac{q^5J_{10,100}^2J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^4J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^4}. \tag{3.4.37}$$

This follows from the two identities

$$\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^4J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{45,100}^4} - \frac{4q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}$$
$$= \frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{J_{100}^9} \tag{3.4.38}$$

and

$$\frac{J_{10,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^5J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{45,100}^5}-\frac{q^5J_{10,100}^2J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^4J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^4}$$
$$=\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^4J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{45,100}^4}. \tag{3.4.39}$$

We obtain (3.4.38) after multiplying by $\frac{J_{15,100}J_{35,100}J_{50,100}^2J_{100}^{15}}{q^5J_{5,100}J_{30,100}J_{45,100}}$ on both sides of (3.4.13).

Next, we prove (3.4.39). Noting that $\frac{J_{k,50}}{J_{50}}=\frac{J_{k,100}J_{50-k,100}}{J_{100}^2}$, we find that (3.2.4) is equivalent to

$$J_{15,100}J_{20,100}J_{30,100}J_{35,100}-q^5J_{5,100}J_{10,100}J_{40,100}J_{45,100}=\frac{J_{5,100}J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{45,100}}{J_{10,100}J_{15,100}J_{35,100}J_{40,100}}. \tag{3.4.40}$$

Multiplying by $\frac{J_{10,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^5J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^5}$ on both sides of the above equation, we recover (3.4.39). This completes our proof of (3.4.28).

To prove (3.4.29), it suffices to show that

$$-\frac{J_{5,100}J_{10,100}^2J_{15,100}J_{30,100}^2J_{35,100}J_{40,100}^2J_{45,100}J_{50,100}}{J_{100}^9}$$
$$=\left\{\frac{2q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^2J_{25,100}^4J_{35,100}^2J_{40,100}^2J_{45,100}^2}-\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^3J_{20,100}J_{25,100}^2J_{35,100}^3J_{40,100}J_{45,100}^2}\right\}$$
$$+\left\{\frac{2q^5J_{10,100}J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}^2J_{20,100}^3J_{25,100}^4J_{30,100}^2J_{35,100}^2J_{45,100}^2}-\frac{q^5J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^4J_{35,100}^4J_{40,100}^2J_{45,100}^2}\right\}. \tag{3.4.41}$$

This follows from the three identities

$$\frac{2q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^2J_{25,100}^4J_{35,100}^2J_{40,100}^2J_{45,100}^2}$$
$$-\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^3J_{20,100}J_{25,100}^2J_{35,100}^3J_{40,100}J_{45,100}^2}$$

$$= -\frac{J_{30,100}J_{50,100}^2J_{100}^3}{J_{15,100}J_{25,100}^2J_{35,100}}, \tag{3.4.42}$$

$$\frac{2q^5J_{10,100}J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}^2J_{20,100}^3J_{25,100}^4J_{30,100}^2J_{35,100}^2J_{45,100}^2} - \frac{q^5J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^4J_{35,100}^4J_{40,100}^2J_{45,100}^2}$$
$$= \frac{q^5J_{10,100}^3J_{40,100}^2J_{50,100}J_{100}^3}{J_{5,100}^2J_{20,100}^2J_{30,100}J_{45,100}^2} \tag{3.4.43}$$

and

$$-\frac{J_{5,100}J_{10,100}^2J_{15,100}J_{30,100}^2J_{35,100}J_{40,100}^2J_{45,100}J_{50,100}}{J_{100}^9}$$
$$= \frac{q^5J_{10,100}^3J_{40,100}^2J_{50,100}J_{100}^3}{J_{5,100}^2J_{20,100}^2J_{30,100}J_{45,100}^2} - \frac{J_{30,100}J_{50,100}^2J_{100}^3}{J_{15,100}J_{25,100}^2J_{35,100}}. \tag{3.4.44}$$

By substituting (3.4.42) and (3.4.43) into (3.4.44), we can recover (3.4.41). Hence it suffices to prove (3.4.42), (3.4.43) and (3.4.44).

First, we prove (3.4.42). Multiplying by $\frac{J_{5,100}J_{10,100}J_{15,100}^2J_{20,100}J_{25,100}^2J_{35,100}^2J_{40,100}J_{45,100}}{J_{50,100}^2J_{100}^{11}}$ throughout and noting that

$$\frac{2q^5J_{20,100}J_{100}^4}{J_{5,100}J_{25,100}^2J_{40,100}J_{45,100}} = \frac{q^5\overline{J}_{0,50}\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{25,50}}{J_{50}^4},$$
$$-\frac{J_{100}^4}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}} = -\frac{\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50}}{J_{50}^4}$$

and

$$-\frac{J_{5,100}J_{10,100}J_{15,100}J_{20,100}J_{30,100}J_{35,100}J_{40,100}J_{45,100}}{J_{100}^8} = -\frac{J_{5,50}J_{10,50}J_{15,50}J_{20,50}}{J_{50}^4},$$

we find that (3.4.42) is equivalent to

$$q^5\overline{J}_{0,50}\overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{25,50} - \overline{J}_{5,50}\overline{J}_{10,50}\overline{J}_{15,50}\overline{J}_{20,50} = -J_{5,50}J_{10,50}J_{15,50}J_{20,50}.$$

This follows from (3.2.12).

Next, we prove (3.4.43). Multiplying by $\frac{J_{5,100}^2 J_{30,100} J_{45,100}^2}{q^5 J_{10,100} J_{50,100} J_{100}^5}$ throughout and noting that

$$\frac{2J_{40,100}J_{50,100}J_{10,100}^{10}}{J_{15,100}^2 J_{20,100}^3 J_{25,100}^4 J_{30,100} J_{35,100}^2} = \frac{\overline{J}_{15,50}^2 \overline{J}_{20,50} J_{50}^2}{J_{20,50}^2 J_{15,50}^2},$$

$$-\frac{J_{30,100}^2 J_{10,100}^{10}}{J_{10,100}^2 J_{15,100}^4 J_{35,100}^4 J_{40,100}^2} = -\frac{\overline{J}_{15,50}^2 J_{50}^2}{J_{10,50}^2 J_{15,50}^2}$$

and

$$\frac{J_{10,100}^2 J_{40,100}^2}{J_{20,100}^2 J_{100}^2} = \frac{J_{50}^2}{\overline{J}_{10,50}^2},$$

we find that (3.4.43) is equivalent to

$$\frac{\overline{J}_{20,50}\overline{J}_{25,50}^2\overline{J}_{0,50}}{\overline{J}_{10,50}^2 J_{10,50}^2 J_{15,50}^2} - \frac{\overline{J}_{15,50}^2}{J_{10,50}^2 J_{15,50}^2} = \frac{1}{\overline{J}_{10,50}^2}.$$

This follows from dividing by $J_{10,50}^2 J_{15,50}^2 \overline{J}_{10,50}^2$ on both sides of (3.2.19).

Finally, we prove (3.4.44). Multiplying by $\frac{1}{J_{50,100}J_{100}}$ throughout and noting that

$$-\frac{J_{5,100}J_{10,100}^2 J_{15,100} J_{30,100}^2 J_{35,100} J_{40,100}^2 J_{45,100}}{J_{100}^{10}} = -\frac{J_{5,50}J_{15,50}J_{20,50}^2}{\overline{J}_{10,50}^2 J_{50}^2},$$

$$\frac{q^5 J_{10,100}^3 J_{40,100}^2 J_{100}^2}{J_{5,100}^2 J_{20,100}^2 J_{30,100} J_{45,100}^2} = \frac{q^5 \overline{J}_{5,50}^2 \overline{J}_{20,50}}{\overline{J}_{10,50} J_{50}^2}$$

and

$$-\frac{J_{30,100}J_{50,100}J_{100}^2}{J_{15,100}J_{25,100}^2 J_{35,100}} = -\frac{\overline{J}_{15,50}\overline{J}_{25,50}}{J_{50}^2},$$

we find that (3.4.44) is equivalent to

$$-\frac{J_{5,50}J_{15,50}J_{20,50}^2}{\overline{J}_{10,50}^2} = \frac{q^5 \overline{J}_{5,50}^2 \overline{J}_{20,50}}{\overline{J}_{10,50}} - \overline{J}_{15,50}\overline{J}_{25,50}.$$

Multiplying by $\overline{J}_{10,50}^2$ throughout and rearranging, the above equation becomes

$$\overline{J}_{10,50}^2 \overline{J}_{15,50} \overline{J}_{25,50} - J_{5,50} J_{15,50} J_{20,50}^2 = q^5 \overline{J}_{5,50}^2 \overline{J}_{10,50} \overline{J}_{20,50}.$$

The above equation follows from replacing q by q^{50} and setting

$$(A, b, c, d, e) = (-q^{30}, q^{20}, q^{20}, q^5, q^{15})$$

in (3.2.11). This completes the proof of (3.4.29).

To prove (3.4.30), it suffices to show that

$$\begin{aligned}
&\frac{J_{10,100}^3 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^3 J_{50,100}^2}{2J_{20,100} J_{100}^9} \\
=&\frac{2q^{10} J_{10,100} J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{15,100} J_{20,100}^2 J_{25,100}^4 J_{30,100}^2 J_{35,100} J_{45,100}^3} + \frac{J_{10,100} J_{15,100}^3 J_{20,100}^2 J_{30,100} J_{35,100}^3 J_{50,100}^3}{2J_{25,100}^2 J_{100}^9} \\
&- \frac{2q^{10} J_{50,100}^2 J_{100}^{15}}{J_{5,100} J_{15,100}^3 J_{20,100}^2 J_{25,100}^4 J_{30,100} J_{35,100}^3 J_{45,100}} - \frac{q^{10} J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{30,100} J_{35,100}^3 J_{45,100}^3}.
\end{aligned} \tag{3.4.45}$$

Combining the second product on the right side with the one on the left side, the first product with the third one on the right side, respectively, we find that the above equation follows from the three identities

$$\begin{aligned}
&\frac{J_{10,100}^3 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^3 J_{50,100}^2}{2J_{20,100} J_{100}^9} - \frac{J_{10,100} J_{15,100}^3 J_{20,100}^2 J_{30,100} J_{35,100}^3 J_{50,100}^3}{2J_{25,100}^2 J_{100}^9} \\
=& -\frac{q^{10} J_{5,100} J_{10,100}^3 J_{15,100} J_{30,100} J_{35,100} J_{40,100}^2 J_{45,100} J_{50,100}}{J_{100}^9},
\end{aligned} \tag{3.4.46}$$

$$\begin{aligned}
&\frac{2q^{10} J_{10,100} J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{15,100} J_{20,100}^2 J_{25,100}^4 J_{30,100}^2 J_{35,100} J_{45,100}^3} \\
&- \frac{2q^{10} J_{50,100}^2 J_{100}^{15}}{J_{5,100} J_{15,100}^3 J_{20,100}^2 J_{25,100}^4 J_{30,100} J_{35,100}^3 J_{45,100}} \\
=& \frac{4q^{15} J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{15,100}^2 J_{20,100}^2 J_{25,100}^4 J_{30,100} J_{35,100}^2 J_{45,100}^2},
\end{aligned} \tag{3.4.47}$$

and

$$-\frac{q^{10} J_{5,100} J_{10,100}^3 J_{15,100} J_{30,100} J_{35,100} J_{40,100}^2 J_{45,100} J_{50,100}}{J_{100}^9}$$

$$= \frac{4q^{15} J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{15,100}^2 J_{20,100}^2 J_{25,100}^4 J_{30,100} J_{35,100}^2 J_{45,100}^2} - \frac{q^{10} J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{30,100} J_{35,100}^3 J_{45,100}^3}. \tag{3.4.48}$$

By substituting (3.4.46) and (3.4.47) into (3.4.48), we can recover (3.4.45). Hence it suffices to prove (3.4.46), (3.4.47) and (3.4.48).

First, we prove (3.4.46). Multiplying by $-\frac{2J_{100}^7}{J_{10,100}^2 J_{15,100}^2 J_{35,100}^2 J_{40,100} J_{50,100}^2}$ throughout and noting that

$$-\frac{J_{10,100} J_{25,100}^2 J_{40,100}^2}{J_{15,100} J_{20,100} J_{35,100} J_{100}^2} = -\frac{J_{10,50} \overline{J}_{15,50} \overline{J}_{20,50} J_{25,50}}{J_{50}^4},$$
$$\frac{J_{15,100} J_{20,100}^2 J_{30,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100}^2 J_{40,100} J_{100}^2} = \frac{\overline{J}_{10,50} J_{15,50} J_{20,50} \overline{J}_{25,50}}{J_{50}^4}$$

and

$$\frac{2q^{10} J_{5,100} J_{10,100} J_{30,100} J_{40,100} J_{45,100}}{J_{15,100} J_{35,100} J_{50,100} J_{100}^2} = \frac{q^{10} \overline{J}_{0,50} J_{5,50} J_{10,50} \overline{J}_{15,50}}{J_{50}^4},$$

we find that (3.4.46) is equivalent

$$-J_{10,50} \overline{J}_{15,50} \overline{J}_{20,50} J_{25,50} + \overline{J}_{10,50} J_{15,50} J_{20,50} \overline{J}_{25,50} = q^{10} \overline{J}_{0,50} J_{5,50} J_{10,50} \overline{J}_{15,50}$$

This follows from replacing q by q^{50} and setting $(A, b, c, d, e) = (q^{40}, q^{10}, -q^{15}, -q^{30}, q^{25})$ in (3.2.11).

Next, we prove (3.4.47). Multiplying by $\frac{J_{5,100} J_{15,100} J_{20,100}^2 J_{25,100}^4 J_{30,100}^2 J_{35,100} J_{45,100}}{2q^{10} J_{50,100}^2 J_{100}^{12}}$ throughout and noting that

$$\frac{J_{10,100} J_{100}^3}{J_{5,100}^2 J_{45,100}^2} = \frac{\overline{J}_{5,100} \overline{J}_{45,100}}{J_{5,100} J_{45,100}},$$
$$-\frac{J_{30,100} J_{100}^3}{J_{15,100}^2 J_{35,100}^2} = -\frac{\overline{J}_{15,100} \overline{J}_{35,100}}{J_{15,100} J_{35,100}}$$

and

$$\frac{2q^5 J_{10,100} J_{30,100} J_{100}^3}{J_{5,100} J_{15,100} J_{35,100} J_{45,100} J_{50,100}} = \frac{q^5 \overline{J}_{0,100} \overline{J}_{50,100} J_{10,100} J_{30,100}}{J_{5,100} J_{15,100} J_{35,100} J_{45,100}},$$

we find that (3.4.44) is equivalent to

$$\frac{\overline{J}_{5,100}\overline{J}_{45,100}}{J_{5,100}J_{45,100}} - \frac{\overline{J}_{15,100}\overline{J}_{35,100}}{J_{15,100}J_{35,100}} = \frac{q^5\overline{J}_{0,100}\overline{J}_{50,100}J_{10,100}J_{30,100}}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}}.$$

Multiplying by $J_{5,100}J_{15,100}J_{35,100}J_{45,100}$ throughout and rearranging, the above equation becomes

$$J_{15,100}J_{35,100}\overline{J}_{5,100}\overline{J}_{45,100} - J_{5,100}J_{45,100}\overline{J}_{15,100}\overline{J}_{35,100} = q^5\overline{J}_{0,100}\overline{J}_{50,100}J_{10,100}J_{30,100}.$$

This follows from replacing q by q^{100} and setting

$$(A, b, c, d, e) = (-q^{50}, q^5, -q^{15}, -q^{35}, q^{45})$$

in (3.2.11).

Finally, we prove (3.4.48). Multiplying by $\frac{J_{5,100}J_{15,100}J_{20,100}^2J_{30,100}J_{35,100}J_{45,100}}{q^{10}J_{10,100}J_{50,100}J_{100}^7}$ throughout and noting that

$$-\frac{J_{5,100}^2J_{10,100}^2J_{15,100}^2J_{20,100}^2J_{30,100}^2J_{35,100}^2J_{40,100}^2J_{45,100}^2}{J_{100}^{16}} = -\frac{J_{5,25}^2J_{10,25}^2}{J_{25}^4},$$

$$\frac{4q^5J_{100}^8}{J_{5,100}J_{15,100}J_{25,100}^4J_{35,100}J_{45,100}} = \frac{q^5\overline{J}_{0,25}^2\overline{J}_{5,25}\overline{J}_{10,25}}{J_{25}^4}$$

and

$$-\frac{J_{100}^8}{J_{5,100}^2J_{15,100}^2J_{35,100}^2J_{45,100}^2} = -\frac{\overline{J}_{5,25}^2\overline{J}_{10,25}^2}{J_{25}^4},$$

we find that (3.4.48) is equivalent to

$$-J_{5,25}^2J_{10,25}^2 = q^5\overline{J}_{0,25}^2\overline{J}_{5,25}\overline{J}_{10,25} - \overline{J}_{5,25}^2\overline{J}_{10,25}^2.$$

The above equation is equivalent to (3.2.13). This completes the proof of (3.4.30).

To prove (3.4.31), it suffices to show that

$$-\frac{J_{5,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{45,100}J_{50,100}^2}{2J_{40,100}J_{100}^9}$$

$$=\frac{q^5 J_{20,100}J_{30,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^3J_{25,100}^2J_{35,100}^3J_{40,100}^3J_{45,100}^2}$$
$$-\frac{2q^{10}J_{10,100}^2J_{40,100}^2J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}^2J_{20,100}^4J_{25,100}^4J_{30,100}^3J_{35,100}^2J_{45,100}^2}$$
$$-\frac{J_{5,100}J_{10,100}^2J_{15,100}^2J_{35,100}^2J_{40,100}^2J_{45,100}J_{50,100}^3}{2J_{25,100}^2J_{100}^9}-\frac{q^5 J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^3J_{35,100}^3J_{40,100}^2J_{45,100}^3}$$
$$+\frac{2q^{10}J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^2J_{30,100}J_{35,100}^2J_{45,100}^3}. \tag{3.4.49}$$

Combining the third product on the right side with the one on the left side, the first product with the fourth and the second product with the last one on the right side, respectively, we find that the above equation follows from the four identities

$$\frac{J_{5,100}J_{10,100}^2J_{15,100}^2J_{35,100}^2J_{40,100}^2J_{45,100}J_{50,100}^3}{2J_{25,100}^2J_{100}^9}-\frac{J_{5,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{45,100}J_{50,100}^2}{2J_{40,100}J_{100}^9}$$
$$=-\frac{q^{10}J_{10,100}^3J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}}{J_{100}^9}, \tag{3.4.50}$$

$$\frac{q^5 J_{20,100}J_{30,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^3J_{25,100}^2J_{35,100}^3J_{40,100}^3J_{45,100}^2}-\frac{q^5 J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^3J_{35,100}^3J_{40,100}^2J_{45,100}^3}$$
$$=-\frac{q^{10}J_{20,100}^2J_{30,100}^2J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^4J_{35,100}^4J_{40,100}^4J_{45,100}^2}, \tag{3.4.51}$$

$$\frac{2q^{10}J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^2J_{30,100}J_{35,100}^2J_{45,100}^3}-\frac{2q^{10}J_{10,100}^2J_{40,100}^2J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}^2J_{20,100}^4J_{25,100}^4J_{30,100}^3J_{35,100}^2J_{45,100}^2}$$
$$=\frac{2q^{15}J_{10,100}^3J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^4J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^4}, \tag{3.4.52}$$

$$-\frac{q^{10}J_{10,100}^3J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}}{J_{100}^9}$$
$$=\frac{2q^{15}J_{10,100}^3J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^4J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^4}-\frac{q^{10}J_{20,100}^2J_{30,100}^2J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^4J_{35,100}^4J_{40,100}^4J_{45,100}^2}. \tag{3.4.53}$$

By substituting (3.4.50), (3.4.51) and (3.4.52) into (3.4.53), we can recover (3.4.49). Hence it suffices to prove (3.4.50), (3.4.51), (3.4.52) and (3.4.53).

First, we prove (3.4.50). Multiplying by $-\frac{2J_{100}^7}{J_{5,100}J_{10,100}J_{15,100}J_{20,100}J_{30,100}J_{35,100}J_{45,100}J_{50,100}^2}$

throughout and noting that

$$-\frac{J_{10,100}J_{15,100}J_{35,100}J_{40,100}^2J_{50,100}}{J_{20,100}J_{25,100}^2J_{30,100}J_{100}^2} = -\frac{J_{10,50}J_{15,50}\overline{J}_{20,50}\overline{J}_{25,50}}{J_{50}^4},$$
$$\frac{J_{20,100}^2J_{25,100}^2J_{30,100}^2}{J_{10,100}J_{15,100}J_{35,100}J_{40,100}J_{100}^2} = \frac{\overline{J}_{10,50}\overline{J}_{15,50}J_{20,50}J_{25,50}}{J_{50}^4}$$

and

$$\frac{q^{10}J_{10,100}^2J_{15,100}J_{35,100}J_{40,100}}{J_{5,100}J_{45,100}J_{50,100}J_{100}^2} = \frac{q^{10}\overline{J}_{0,50}\overline{J}_{5,50}J_{10,50}J_{15,50}}{J_{50}^4},$$

we find that (3.4.50) is equivalent to

$$-J_{10,50}J_{15,50}\overline{J}_{20,50}\overline{J}_{25,50} + \overline{J}_{10,50}\overline{J}_{15,50}J_{20,50}J_{25,50} = q^{10}\overline{J}_{0,50}\overline{J}_{5,50}J_{10,50}J_{15,50}.$$

This follows from replacing q by q^{50} and setting $(A, b, c, d, e) = (q^{40}, q^{10}, q^{15}, -q^{25}, -q^{30})$ in (3.2.11).

Next, we prove (3.4.51). Multiplying by $-\frac{J_{5,100}^3J_{10,100}^2J_{15,100}^3J_{35,100}^3J_{40,100}^4J_{45,100}^3}{q^5J_{20,100}J_{30,100}J_{50,100}J_{100}^{17}}$ throughout and noting that

$$-\frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{25,100}^2J_{100}^2} = -\frac{J_{5,50}J_{20,50}\overline{J}_{20,50}\overline{J}_{25,50}}{J_{50}^4},$$
$$\frac{J_{10,100}J_{40,100}^2}{J_{20,100}J_{100}^2} = \frac{J_{10,50}J_{15,50}\overline{J}_{15,50}\overline{J}_{20,50}}{J_{50}^4}$$

and

$$\frac{q^5J_{5,100}J_{20,100}J_{30,100}J_{45,100}}{J_{15,100}J_{35,100}J_{100}^2} = \frac{q^5J_{5,50}J_{10,50}\overline{J}_{10,50}\overline{J}_{15,50}}{J_{50}^4},$$

we find that (3.4.51) is equivalent to

$$-J_{5,50}J_{20,50}\overline{J}_{20,50}\overline{J}_{25,50} + J_{10,50}J_{15,50}\overline{J}_{15,50}\overline{J}_{20,50} = q^5J_{5,50}J_{10,50}\overline{J}_{10,50}\overline{J}_{15,50}.$$

This follows from replacing q by q^{50} and setting $(A,b,c,d,e) = (q^{45}, q^5, -q^{25}, q^{30}, -q^{30})$ in (3.2.11).

Next, we prove (3.4.52). Multiplying by $\frac{J_{5,100}^3 J_{15,100}^2 J_{20,100}^4 J_{25,100}^2 J_{30,100}^3 J_{35,100}^2 J_{45,100}^3}{2q^{10} J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{17}}$ throughout and noting that

$$\frac{J_{20,100}^2 J_{30,100}^2}{J_{10,100} J_{40,100} J_{100}^2} = \frac{J_{15,50} J_{20,50} \overline{J}_{10,50} \overline{J}_{15,50}}{J_{50}^4},$$
$$-\frac{J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{25,100}^2 J_{100}^2} = -\frac{J_{5,50} J_{20,50} \overline{J}_{20,50} \overline{J}_{25,50}}{J_{50}^4}$$

and

$$\frac{q^5 J_{10,100} J_{15,100} J_{20,100} J_{35,100}}{J_{5,100} J_{45,100} J_{100}^2} = \frac{q^5 J_{10,50} J_{15,50} \overline{J}_{5,50} \overline{J}_{10,50}}{J_{50}^4},$$

we find that (3.4.52) is equivalent to

$$J_{15,50} J_{20,50} \overline{J}_{10,50} \overline{J}_{15,50} - J_{5,50} J_{20,50} \overline{J}_{20,50} \overline{J}_{25,50} = q^5 J_{10,50} J_{15,50} \overline{J}_{5,50} \overline{J}_{10,50}.$$

This follows from replacing q by q^{50} and setting $(A,b,c,d,e) = (q^{35}, q^5, q^{20}, -q^{20}, -q^{25})$ in (3.2.11).

Finally, we prove (3.4.53). Multiplying by $\frac{J_{5,100} J_{15,100} J_{35,100} J_{40,100}^2 J_{45,100}}{q^{10} J_{50,100} J_{100}^7}$ throughout and noting that

$$-\frac{J_{5,100} J_{10,100}^3 J_{15,100}^3 J_{20,100} J_{30,100} J_{35,100}^3 J_{40,100}^3 J_{45,100}}{J_{100}^{16}} = -\frac{J_{5,25} J_{10,25}^3}{J_{25}^4},$$
$$\frac{2q^5 J_{10,100}^3 J_{40,100}^3 J_{100}^8}{J_{5,100}^3 J_{20,100}^3 J_{25,100}^2 J_{30,100}^3 J_{45,100}^3} = \frac{q^5 \overline{J}_{5,25}^3 \overline{J}_{0,25}}{J_{25}^4}$$

and

$$-\frac{J_{20,100}^2 J_{30,100}^2 J_{100}^8}{J_{5,100} J_{10,100}^2 J_{15,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}} = -\frac{\overline{J}_{5,25} \overline{J}_{10,25}^3}{J_{25}^4},$$

we find that (3.4.53) is equivalent to

$$-J_{5,25}J_{10,25}^3 = q^5\overline{J}_{5,25}^3\overline{J}_{25,25} - \overline{J}_{5,25}\overline{J}_{10,25}^3,$$

which follows from replacing q by q^{25} and setting

$$(A, b, c, d, e) = (-q^{25}, -q^5, -q^{15}, -q^{15}, -q^{15})$$

in (3.2.11). This completes the proof of (3.4.31).

To prove (3.4.32), it suffices to show that

$$\begin{aligned} 0 = &\frac{J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100} J_{25,100}^2 J_{30,100} J_{35,100}^2 J_{40,100} J_{45,100}^3} \\ &- \frac{2q^5 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100} J_{25,100}^2 J_{35,100}^3 J_{40,100} J_{45,100}^2} \\ &- \frac{J_{50,100}^2 J_{100}^{15}}{J_{5,100} J_{10,100} J_{15,100}^4 J_{20,100} J_{25,100}^2 J_{35,100}^4 J_{40,100} J_{45,100}}, \end{aligned}$$

which follows from dividing by $\frac{J_{50,100}^2 J_{100}^{15}}{J_{5,100}J_{10,100}J_{15,100}^2 J_{20,100}J_{25,100}^2 J_{30,100}J_{35,100}^2 J_{40,100}J_{45,100}}$ on both sides of (3.2.6). □

Now we are in a position to prove Theorem 3.1.2.

Proof of Theorem 3.1.2. First we prove (3.1.5) and (3.1.7). Replacing z by $\xi_{10} = e^{\frac{\pi i}{5}}$ in (3.1.1), we have

$$L_{10}(q) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_2(m,n)\xi_{10}^m q^n = \sum_{n=0}^{\infty}\sum_{m=0}^{9}\sum_{j=-\infty}^{\infty} N_2(10j+m,n)\xi_{10}^m q^n.$$

By the definition of $N_2(m, 10, n)$, we know that

$$L_{10}(q) = \sum_{n=0}^{\infty}\sum_{m=0}^{9} N_2(m,10,n)\xi_{10}^m q^n = \sum_{m=0}^{9}\sum_{n=0}^{\infty} N_2(m,10,n)\xi_{10}^m q^n.$$

Expanding the last series in the above equation according to the summation index m, noting that $N_2(m,10,n) = N_2(10-m,10,n), \xi_{10}^5 = -1$ and $\xi_{10} + \xi_{10}^3 + \xi_{10}^7 + \xi_{10}^9 = 1$, we have

$$L_{10}(q) = \sum_{n\geq 0} \left(N_2(0,10,n) + N_2(1,10,n) - N_2(4,10,n) - N_2(5,10,n)\right) q^n$$
$$+ (\xi_{10}^2 - \xi_{10}^3) \sum_{n\geq 0} \left(N_2(1,10,n) + N_2(2,10,n) - N_2(3,10,n) - N_2(4,10,n)\right) q^n. \tag{3.4.54}$$

Next, replacing z by ξ_{10} in (3.3.14) and simplifying, we find that

$$\begin{aligned}
&L_{10}(q)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-\frac{1}{\xi_{10}})(1-\xi_{10})q^{2n^2+n}}{(1-\xi_{10}q^{2n})(1-q^{2n}/\xi_{10})}\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+\xi_{10}^4)(1+\xi_{10}^6)q^{2n^2+n}}{(1+\xi_{10}^6q^{2n})(1+\xi_{10}^4q^{2n})}\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+\xi_{10}^4)(1+\xi_{10}^6)q^{2n^2+n}(1+q^{2n})(1+\xi_{10}^2q^{2n})(1+\xi_{10}^8q^{2n})}{(1+\xi_{10}^6q^{2n})(1+\xi_{10}^4q^{2n})(1+q^{2n})(1+\xi_{10}^2q^{2n})(1+\xi_{10}^8q^{2n})}\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^nq^{2n^2+n}(1+q^{2n})}{1+q^{10n}}\\
&\quad + \frac{(\xi_{10}^2-\xi_{10}^3)(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^nq^{2n^2+n}(q^{4n}-1)}{1+q^{10n}}\\
&= F_3(q) + (\xi_{10}^2-\xi_{10}^3)F_4(q).
\end{aligned}$$

By the above equation and (3.4.54), we have

$$\sum_{n\geq 0} \left(N_2(0,10,n) + N_2(1,10,n) - N_2(4,10,n) - N_2(5,10,n)\right) q^n + (\xi_{10}^2 - \xi_{10}^3)\times$$
$$\sum_{n\geq 0} \left(N_2(1,10,n) + N_2(2,10,n) - N_2(3,10,n) - N_2(4,10,n)\right) q^n$$
$$= F_3(q) + (\xi_{10}^2 - \xi_{10}^3)F_4(q). \tag{3.4.55}$$

Since the coefficients of $F_3(q)$ and $F_4(q)$ are all integers and $[\mathbf{Q}(\xi_{10}):\mathbf{Q}]=4$, we equate the coefficient of ξ_{10}^k on both sides of (3.4.55) and find that

$$\sum_{n=0}^{\infty}(N_2(0,10,n)+N_2(1,10,n)-N_2(4,10,n)-N_2(5,10,n))q^n=F_3(q)$$

and

$$\sum_{n=0}^{\infty}(N_2(1,10,n)+N_2(2,10,n)-N_2(3,10,n)-N_2(4,10,n))q^n=F_4(q).$$

This completes the proofs of (3.1.5) and (3.1.7).

Next, we prove (3.1.6) and (3.1.8). Since we have

$$F_3(q)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{10n}},$$

substituting (3.4.1) and (3.4.2) into the series on the right side of the above equation, we find

$$\begin{aligned}F_3(q)&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_0+\frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\right\}\\&\quad+\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_1+\frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+75n+24}}{1+q^{50n+30}}\right\}\\&=\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}+\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+75n+24}}{1+q^{50n+30}}\\&\quad+\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(P_0+P_1).\end{aligned}\tag{3.4.56}$$

We complete our proof of (3.1.6) after invoking Lemma 3.4.2. Similarly, since we have

$$F_4(q)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(q^{4n}-1)}{1+q^{10n}},$$

substituting (3.4.2) and (3.4.3) into the series on the right side of the above equation, we find

$$\begin{aligned}F_4(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_2 - \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n-3}}{1+q^{50n}}\right\}\\ &\quad - \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_0 + \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\right\}\\ &= -\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n-3}}{1+q^{50n}} - \frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\\ &\quad + \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(P_2 - P_0). \end{aligned} \tag{3.4.57}$$

We complete our proof after invoking Lemma 3.4.3. □

3.5 Proof of the inequalities

Proof. First we prove (3.1.9) and (3.1.10). By [56, Theorem 1.1], we have

$$\begin{aligned}&\sum_{n\geq 0}(N_2(0,3,3n+1) - N_2(1,3,3n+1))q^n\\ &\quad = \frac{(-q^3,q^6;q^6)_\infty}{(q^2,q^4;q^6)_\infty}\\ &\quad = \frac{(-q^3;q^6)_\infty}{(q^2;q^6)_\infty} \times \sum_{n\geq 0}\frac{q^{4n}(q^2;q^6)_n}{(q^6;q^6)_n} \quad \text{(by q-binomial theorem)}\\ &\quad = (-q^3;q^6)_\infty \times \sum_{n\geq 0}\frac{q^{4n}}{(q^{6n+2};q^6)_\infty(q^6;q^6)_n}.\end{aligned}$$

It is clear that a product of terms of the type $\frac{1}{1-q^m}$ has nonnegative coefficients and the term $\frac{1+q^3}{1-q^2}$ appearing in the last expression has positive coefficients of q^n for all $n \geq 2$. Inequality (3.1.9) follows.

Similarly, we have

$$\sum_{n\geq 0}(N_2(0,3,3n+2) - N_2(1,3,3n+2))q^n$$

$$= \frac{(q^3;q^3)_\infty(-q^6;q^6)_\infty}{(q,q^5;q^6)_\infty(q^4,q^8;q^{12})_\infty}$$
$$= \frac{(q^3;q^3)_\infty}{(q,q^5;q^6)_\infty} \times \frac{(q^2,q^4;q^6)_\infty}{(q^2,q^4;q^6)_\infty} \times \frac{(-q^6;q^6)_\infty}{(q^4,q^8;q^{12})_\infty}$$
$$= \frac{(q^3;q^3)_\infty(q^2,q^4;q^6)_\infty}{(q,q^2;q^3)_\infty} \times \frac{(-q^6;q^6)_\infty}{(q^4,q^8;q^{12})_\infty}$$
$$= \frac{(-q,-q^2,q^3;q^3)_\infty(-q^6;q^6)_\infty}{(q^4,q^8;q^{12})_\infty}$$
$$= \frac{(-q^6;q^6)_\infty \sum_{n=-\infty}^{\infty} q^{\frac{3n^2+n}{2}}}{(q^4,q^8;q^{12})_\infty} \quad \text{(by Jacobi's triple product identity)}\ .$$

Since $\sum_{n=-\infty}^{\infty} q^{\frac{3n^2+n}{2}} = 1+q+q^2+q^5+O(q^5)$ and the factor $1-q^4$ appears in the numerator, (3.1.10) follows.

Next, we prove (3.1.11), (3.1.12) and (3.1.13). By [56, Theorem 1.2], we have

$$\sum_{n=0}^{\infty} \left(N_2(1,5,5n+3) - N_2(2,5,5n+3)\right) q^n$$
$$= \frac{(-q^5,q^{10};q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}$$
$$= \frac{(-q^5;q^{10})_\infty}{(q^4;q^{10})_\infty} \sum_{n=0}^{\infty} \frac{q^{6n}(q^4;q^{10})_n}{(q^{10};q^{10})_n} \quad \text{(by } q\text{-binomial Theorem)}$$
$$= (-q^5;q^{10})_\infty \sum_{n=0}^{\infty} \frac{q^{6n}}{(q^{4+10n};q^{10})_\infty(q^{10};q^{10})_n}.$$

Since the factor $\frac{1+q^5}{1-q^4}$ appears in the last expression, (3.1.11) follows.

Similarly, we have

$$\sum_{n=0}^{\infty} \left(N_2(0,5,5n+1) - N_2(1,5,5n+1)\right) q^n$$
$$= \sum_{n=0}^{\infty} \left\{\left(N_2(0,5,5n+1) - N_2(2,5,5n+1)\right) - \left(N_2(1,5,5n+1) - N_2(2,5,5n+1)\right)\right\} q^n$$
$$= \frac{(-q^5,q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty} - 0 \quad \text{(by [56, Theorem 1.2])}$$
$$= \frac{(-q^5;q^{10})_\infty}{(q^2;q^{10})_\infty} \sum_{n=0}^{\infty} \frac{q^{8n}(q^2;q^{10})_n}{(q^{10};q^{10})_n}$$

$$= (-q^5;q^{10})_\infty \sum_{n=0}^{\infty} \frac{q^{8n}}{(q^{2+10n};q^{10})_\infty (q^{10};q^{10})_n}.$$

Since the factor $\frac{1+q^5}{1-q^2}$ appears in the last expression, (3.1.12) follows.

Finally, since we have

$$\sum_{n=0}^{\infty} \left(N_2(0,5,5n+3) - N_2(1,5,5n+3)\right) q^n$$
$$= \sum_{n=0}^{\infty} \left\{ \left(N_2(0,5,5n+3) - N_2(2,5,5n+3)\right) - \left(N_2(1,5,5n+3) - N_2(2,5,5n+3)\right) \right\} q^n$$
$$= 0 - \sum_{n=0}^{\infty} \left(N_2(1,5,5n+3) - N_2(2,5,5n+3)\right) q^n \quad \text{(by [56, Theorem 1.2])},$$

inequality (3.1.13) follows from (3.1.11).

Next, we prove (3.1.14) and (3.1.15). By (3.1.4), we have

$$\begin{aligned}
&\sum_{n\geq 0} (N_2(0,6,3n) + N_2(1,6,3n) - N_2(2,6,3n) - N_2(3,6,3n))q^n \\
&\quad = \frac{J_{2,12}J_{6,12}^2 J_{12}^3}{J_{1,12}J_{3,12}^3 J_{5,12}} - \frac{1}{J_{3,12}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+9n+3}}{1+q^{6n+4}} \\
&\quad = \frac{1}{J_{3,12}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n^2+3n}}{1-q^{6n+1}} \\
&\quad = \sum_{n=0}^{\infty} \frac{(-q^3;q^6)_n q^{6n^2}}{(q;q^6)_{n+1}(q^5;q^6)_n},
\end{aligned}$$

the second equality follows from setting $r = 0, s = 2$ and replacing q, b_1 and b_2 by $q^6, -q^4$ and q, respectively, in (3.2.24) and the last equality follows from replacing q and c by $-q^3$ and $-q$, respectively, in [4, Entry 12.2.1 (p.1)]. Since the factor $1-q$ appears in the numerator, inequality (3.1.14) follows.

Similarly, we have

$$\sum_{n\geq 0} (N_2(0,6,3n+1) + N_2(1,6,3n+1) - N_2(2,6,3n+1) - N_2(3,6,3n+1))q^n$$

$$
\begin{aligned}
&= \frac{J_{2,12}^2 J_{6,12} J_{12}^3}{J_{1,12}^2 J_{3,12} J_{5,12}^2} \\
&= \frac{[q^2, q^2, q^6; q^{12}]_\infty J_{12}}{[q, q, q^3, q^5, q^5; q^{12}]_\infty} \\
&= \frac{[q^2, q^2, q^6; q^{12}]_\infty J_{12}}{[q, q; q^6]_\infty [q^3; q^{12}]_\infty} \\
&= \frac{[-q, -q; q^6]_\infty [q^6; q^{12}]_\infty J_{12}}{[q^3; q^{12}]_\infty} \\
&= \frac{[-q, -q; q^6]_\infty J_6^2}{J_{12}[q^3; q^{12}]_\infty} \\
&= \frac{(\sum_{n=-\infty}^{\infty} q^{3n^2+2n})^2}{J_{12}[q^3; q^{12}]_\infty} \quad \text{(by Jacobi's triple product identity)} .
\end{aligned}
$$

Since $(\sum_{n=-\infty}^{\infty} q^{3n^2+2n})^2 = 1 + 2q + q^2 + O(q^3)$ and the factor $1 - q^3$ appears in the numerator, (3.1.15) follows from the last equality of above equation.

Next, we prove (3.1.16). By (3.1.6), when $i = 1$, we have

$$
\begin{aligned}
&\sum_{n\geq 0} (N_2(0, 10, 5n + 1) + N_2(1, 10, 5n + 1) - N_2(4, 10, 5n + 1) - N_2(5, 10, 5n + 1))q^n \\
&= \frac{J_{4,20} J_{6,20}^2 J_{10,20} J_{20}^{15}}{J_{1,20}^2 J_{2,20}^2 J_{3,20}^4 J_{5,20} J_{7,20}^4 J_{8,20}^3 J_{9,20}^2} \\
&= \frac{[q^4, q^6, q^6, q^{10}; q^{20}]_\infty J_{20}}{[q^5; q^{20}]_\infty [q, q^2, q^9; q^{20}]_\infty^2 [q^8; q^{20}]_\infty^3 [q^3, q^7; q^{20}]_\infty^4} \\
&= \frac{[q^4, q^6, q^6; q^{20}]_\infty (q^{10}; q^{20})_\infty J_{10}}{(q^5; q^{10})_\infty [q, q^2; q^{10}]_\infty^2 [q^8; q^{20}]_\infty [q^3; q^{10}]_\infty^4} \\
&= \frac{[-q^2, -q^3, -q^3; q^{10}]_\infty (-q^5; q^{10})_\infty J_{10}}{[q; q^{10}]_\infty^2 [q^2; q^{10}]_\infty [q^8; q^{20}]_\infty [q^3; q^{10}]_\infty^2} \\
&= \frac{[-q^2, -q^3, -q^3; q^{10}]_\infty (-q^5; q^{10})_\infty}{[q; q^{10}]_\infty^2 [q^8; q^{20}]_\infty [q^3; q^{10}]_\infty^2} \times \frac{J_{10}}{[q^2; q^{10}]_\infty} \\
&= \frac{[-q^2, -q^3, -q^3; q^{10}]_\infty (-q^5; q^{10})_\infty}{[q; q^{10}]_\infty^2 [q^8; q^{20}]_\infty [q^3; q^{10}]_\infty^2} \times \frac{1}{(q^2; q^{10})_\infty} \times \sum_{n=0}^{\infty} \frac{q^{8n}(q^2; q^{10})_n}{(q^{10}; q^{10})_n} \\
&= \frac{[-q^2, -q^3, -q^3; q^{10}]_\infty (-q^5; q^{10})_\infty}{[q; q^{10}]_\infty^2 [q^8; q^{20}]_\infty [q^3; q^{10}]_\infty^2} \times \sum_{n=0}^{\infty} \frac{q^{8n}}{(q^{10}; q^{10})_n (q^{2+10n}; q^{10})_\infty}.
\end{aligned}
$$

Since the factor $\frac{1}{1-q}$ appears in the product of the last expression, for $i = 1$ inequality (3.1.16) follows.

When $i=2$, we have

$$\sum_{n\geq 0}(N_2(0,10,5n+2)+N_2(1,10,5n+2)-N_2(4,10,5n+2)-N_2(5,10,5n+2))q^n$$
$$=\frac{J_{10,20}J_{20}^{15}}{J_{1,20}^3J_{3,20}^3J_{4,20}J_{5,20}J_{7,20}^3J_{8,20}J_{9,20}^3}$$
$$=\frac{[q^{10};q^{20}]_\infty J_{20}}{[q^4,q^5,q^8;q^{20}]_\infty[q,q^3,q^7,q^9;q^{20}]_\infty^3}$$
$$=\frac{(q^{10};q^{20})_\infty J_{10}}{(q^5;q^{10})_\infty[q^4,q^8;q^{20}]_\infty[q,q^3;q^{10}]_\infty^3}$$
$$=\frac{(-q^5;q^{10})_\infty}{[q^4,q^8;q^{20}]_\infty[q^3;q^{10}]_\infty^2[q;q^{10}]_\infty^3}\times\frac{J_{10}}{[q^3;q^{10}]_\infty}$$
$$=\frac{(-q^5;q^{10})_\infty}{[q^4,q^8;q^{20}]_\infty[q^3;q^{10}]_\infty^2[q;q^{10}]_\infty^3}\times\frac{1}{(q^3;q^{10})_\infty}\times\sum_{n=0}^{\infty}\frac{q^{7n}(q^3;q^{10})_n}{(q^{10};q^{10})_n}$$
$$=\frac{(-q^5;q^{10})_\infty}{[q^4,q^8;q^{20}]_\infty[q^3;q^{10}]_\infty^2[q;q^{10}]_\infty^3}\times\sum_{n=0}^{\infty}\frac{q^{7n}}{(q^{10};q^{10})_n(q^{3+10n};q^{10})_\infty}.$$

Since the factor $\frac{1}{1-q}$ appears in the product of the last expression, for $i=2$ inequality (3.1.16) follows.

When $i=3$, we have

$$\sum_{n\geq 0}(N_2(0,10,5n+3)+N_2(1,10,5n+3)-N_2(4,10,5n+3)-N_2(5,10,5n+3))q^n$$
$$=\frac{J_{2,20}^2J_{8,20}J_{10,20}J_{20}^{15}}{J_{1,20}^4J_{3,20}^2J_{4,20}^3J_{5,20}J_{6,20}^2J_{7,20}^2J_{9,20}^4}$$
$$=\frac{[q^2,q^2,q^8,q^{10};q^{20}]_\infty J_{20}}{[q^5;q^{20}]_\infty[q^3,q^6,q^7;q^{20}]_\infty^2[q^4;q^{20}]_\infty^3[q,q^9;q^{20}]_\infty^4}$$
$$=\frac{[q^2,q^2,q^8;q^{20}]_\infty(q^{10};q^{20})_\infty J_{10}}{(q^5;q^{10})_\infty[q^3,q^4;q^{10}]_\infty^2[q^4;q^{20}]_\infty[q;q^{10}]_\infty^4}$$
$$=\frac{[-q,-q,-q^4;q^{10}]_\infty(-q^5;q^{10})_\infty J_{10}}{[q^3;q^{10}]_\infty^2[q^4;q^{10}]_\infty[q^4;q^{20}]_\infty[q;q^{10}]_\infty^2}$$
$$=\frac{[-q,-q,-q^4;q^{10}]_\infty(-q^5;q^{10})_\infty}{[q^3;q^{10}]_\infty^2[q^4;q^{10}]_\infty[q^4;q^{20}]_\infty[q;q^{10}]_\infty}\times\frac{J_{10}}{[q;q^{10}]_\infty}$$
$$=\frac{[-q,-q,-q^4;q^{10}]_\infty(-q^5;q^{10})_\infty}{[q^3;q^{10}]_\infty^2[q^4;q^{10}]_\infty[q^4;q^{20}]_\infty[q;q^{10}]_\infty}\times\frac{1}{(q;q^{10})_\infty}\times\sum_{n=0}^{\infty}\frac{q^{9n}(q;q^{10})_n}{(q^{10};q^{10})_n}$$

$$= \frac{[-q,-q,-q^4;q^{10}]_\infty(-q^5;q^{10})_\infty}{[q^3;q^{10}]_\infty^2[q^4;q^{10}]_\infty[q^4;q^{20}]_\infty[q;q^{10}]_\infty} \times \sum_{n=0}^{\infty} \frac{q^{9n}}{(q^{10};q^{10})_n(q^{1+10n};q^{10})_\infty}.$$

Since the factor $\frac{1}{1-q}$ appears in the product of the last expression, our inequality follows. This completes the proof of (3.1.16).

Next, we prove (3.1.17). By (3.1.8), when $j=1$, we have

$$\sum_{n\geq 0}(N_2(1,10,5n+1)+N_2(2,10,5n+1)-N_2(3,10,5n+1)-N_2(4,10,5n+1))q^n$$
$$= \frac{2qJ_{10,20}J_{20}^{15}}{J_{1,20}^2J_{3,20}^3J_{4,20}J_{5,20}^3J_{7,20}^3J_{8,20}J_{9,20}^2}$$
$$= \frac{2q[q^{10};q^{20}]_\infty J_{20}}{[q^4,q^8;q^{20}]_\infty[q,q^9;q^{20}]_\infty^2[q^3,q^5,q^7;q^{10}]_\infty^3}$$
$$= \frac{2q(q^{10};q^{20})_\infty J_{10}}{(q^5;q^{10})_\infty^3[q^4,q^8;q^{20}]_\infty[q;q^{10}]_\infty^2[q^3;q^{10}]_\infty^3}$$
$$= \frac{2q(-q^5;q^{10})_\infty}{(q^5;q^{10})_\infty^2[q^4,q^8;q^{20}]_\infty[q,q^3;q^{10}]_\infty^2} \times \frac{J_{10}}{[q^3;q^{10}]_\infty}$$
$$= \frac{2q(-q^5;q^{10})_\infty}{(q^5;q^{10})_\infty^2[q^4,q^8;q^{20}]_\infty[q,q^3;q^{10}]_\infty^2} \times \frac{1}{(q^3;q^{10})_\infty} \times \sum_{n=0}^{\infty}\frac{q^{7n}(q^3;q^{10})_n}{(q^{10};q^{10})_n}$$
$$= \frac{2q(-q^5;q^{10})_\infty}{(q^5;q^{10})_\infty^2[q^4,q^8;q^{20}]_\infty[q,q^3;q^{10}]_\infty^2} \times \sum_{n=0}^{\infty}\frac{q^{7n}}{(q^{10};q^{10})_n(q^{3+10n};q^{10})_\infty}.$$

Since the factor $\frac{q}{1-q}$ appears in the product of the last expression, for $j=1$ inequality (3.1.17) follows.

When $j=3$, we have

$$\sum_{n\geq 0}(N_2(1,10,5n+3)+N_2(2,10,5n+3)-N_2(3,10,5n+3)-N_2(4,10,5n+3))q^n$$
$$= \frac{J_{6,20}J_{10,20}J_{20}^{15}}{J_{1,20}^2J_{2,20}J_{3,20}^4J_{5,20}J_{7,20}^4J_{8,20}^2J_{9,20}^2}$$
$$= \frac{[q^6,q^{10};q^{20}]_\infty J_{20}}{[q^5;q^{20}]_\infty[q^2;q^{20}]_\infty[q,q^8,q^9;q^{20}]_\infty^2[q^3,q^7;q^{20}]_\infty^4}$$
$$= \frac{[q^6;q^{20}]_\infty(q^{10};q^{20})_\infty J_{10}}{(q^5;q^{10})_\infty[q^2,q^8,q^8;q^{20}]_\infty[q;q^{10}]_\infty^2[q^3;q^{10}]_\infty^4}$$

$$= \frac{[-q^3;q^{10}]_\infty(-q^5;q^{10})_\infty J_{10}}{[q^2,q^8,q^8;q^{20}]_\infty[q;q^{10}]^2_\infty[q^3;q^{10}]^3_\infty}$$
$$= \frac{[-q^3;q^{10}]_\infty(-q^5;q^{10})_\infty J_{10}}{[q^2,q^8,q^8;q^{20}]_\infty[q;q^{10}]_\infty[q^3;q^{10}]^3_\infty} \times \frac{J_{10}}{[q;q^{10}]_\infty}$$
$$= \frac{[-q^3;q^{10}]_\infty(-q^5;q^{10})_\infty J_{10}}{[q^2,q^8,q^8;q^{20}]_\infty[q;q^{10}]_\infty[q^3;q^{10}]^3_\infty} \times \frac{1}{(q;q^{10})_\infty} \times \sum_{n=0}^{\infty} \frac{q^{9n}(q;q^{10})_n}{(q^{10};q^{10})_n}$$
$$= \frac{[-q^3;q^{10}]_\infty(-q^5;q^{10})_\infty J_{10}}{[q^2,q^8,q^8;q^{20}]_\infty[q;q^{10}]_\infty[q^3;q^{10}]^3_\infty} \times \sum_{n=0}^{\infty} \frac{q^{9n}}{(q^{10};q^{10})_n(q^{1+10n};q^{10})_\infty}.$$

Since the factor $\frac{1}{1-q}$ appears in the series of the last expression, for $j=3$ inequality (3.1.17) follows.

When $j=4$, we have

$$\sum_{n\geq 0}(N_2(1,10,5n+4)+N_2(2,10,5n+4)-N_2(3,10,5n+4)-N_2(4,10,5n+4))q^n$$
$$= \frac{J_{2,20}J_{10,20}J_{20}^{15}}{J^3_{1,20}J^3_{3,20}J^2_{4,20}J_{5,20}J_{6,20}J^3_{7,20}J^3_{9,20}}$$
$$= \frac{[q^2,q^{10};q^{20}]_\infty J_{20}}{[q^5,q^6;q^{20}]_\infty[q^4;q^{20}]^2_\infty[q,q^3,q^7,q^9;q^{20}]^3_\infty}$$
$$= \frac{[q^2;q^{20}]_\infty(q^{10};q^{20})_\infty J_{10}}{(q^5;q^{10})_\infty[q^4,q^4,q^6;q^{20}]_\infty[q,q^3;q^{10}]^3_\infty}$$
$$= \frac{[-q;q^{10}]_\infty(-q^5;q^{10})_\infty}{[q^4,q^4,q^6;q^{20}]_\infty[q;q^{10}]_\infty[q^3;q^{10}]^3_\infty} \times \frac{J_{10}}{[q;q^{10}]_\infty}$$
$$= \frac{[-q;q^{10}]_\infty(-q^5;q^{10})_\infty}{[q^4,q^4,q^6;q^{20}]_\infty[q;q^{10}]_\infty[q^3;q^{10}]^3_\infty} \times \frac{1}{(q;q^{10})_\infty} \times \sum_{n=0}^{\infty} \frac{q^{9n}(q;q^{10})_n}{(q^{10};q^{10})_n}$$
$$= \frac{[-q;q^{10}]_\infty(-q^5;q^{10})_\infty}{[q^4,q^4,q^6;q^{20}]_\infty[q;q^{10}]_\infty[q^3;q^{10}]^3_\infty} \times \sum_{n=0}^{\infty} \frac{q^{9n}}{(q^{10};q^{10})_n(q^{1+10n};q^{10})_\infty}.$$

Since the factor $\frac{1}{1-q}$ appears in the series of the last expression, for $j=4$ inequality (3.1.17) follows. This completes the proof of (3.1.17).

Note that, for $0\leq t<l$, we have

$$N_2(t,l,n) = N_2(t,2l,n) + N_2(l-t,2l,n).$$

Substituting the above equation with the suitable t and l into the relations between M_2-ranks modulo 3 and 5, by comparing the results with (3.1.14), (3.1.15), (3.1.16) and (3.1.17), we can prove the rest of inequalities. For example, since we have $N_2(0,3,3n+1) = N_2(0,6,3n+1) + N_2(3,6,3n+1)$ and $N_2(1,3,3n+1) = N_2(1,6,3n+1) + N_2(2,6,3n+1)$, substituting these two equalities into (3.1.9), we find that

$$N_2(0,6,3n+1) + N_2(3,6,3n+1) \geq N_2(1,6,3n+1) + N_2(2,6,3n+1).$$

Adding this to (3.1.14), we get (3.1.18). We omit the proofs of the other inequalities since they are similar to the above. $\square$

Chapter 4

Two congruences for Appell-Lerch sums

4.1 Introduction

Recall from Chapter 1 that an Appell-Lerch sum is a series of the form

$$AL(x,q,z) := \frac{1}{(z,q/z,q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r+1} q^{r(r+1)/2} z^{r+1}}{1-q^r xz},$$

where x and z are nonzero complex numbers such that neither z nor xz is an integral power of q.

Definition 4.1.1. *For any integers m, j, and p such that $1 \le j \le p-1$, we define the integer $a_{m,j,p}$ by*

$$\sum_{n=0}^{\infty} a_{m,j,p}(n) q^n := -AL(q^{mp}, q^p, q^j) = \frac{1}{(q^j, q^{p-j}, q^p; q^p)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{pr(r+1)/2+jr+j}}{1-q^{pr+j+pm}}.$$

There has been much interest in Appell-Lerch sums and their connections to mock theta functions in recent years. See for example, Andrews and Hickerson [7], Hickerson and Mortenson [39], M. Waldherr [67] and Zwegers [71]. In [23], Chan proved the following theorem, which gives a congruence for an infinite family of Appell-Lerch sums.

Theorem 4.1.2. *For any two coprime integers p and j such that $p \geq 2$ and $1 \leq j \leq p-1$, we have*

$$\sum_{n=0}^{\infty} a_{0,j,p}(pn+pj-j^2)q^n = p\frac{(q^p;q^p)_\infty^4}{(q;q)^3(q^j,q^{p-j};q^p)_\infty^2}. \tag{4.1.1}$$

In particular,

$$\sum_{n=0}^{\infty} a_{0,j,p}(pn+pj-j^2)q^n \equiv 0 \pmod{p}.$$

The first objective of this chapter is to give an extension of Theorem 4.1.2.

Theorem 4.1.3. *For any integer m and any two coprime integers p and j such that $p \geq 2$ and $1 \leq j \leq p-1$, we have*

$$\sum_{n=0}^{\infty} a_{m,j,p}(pn+pj-j^2)q^n = (-1)^m p\frac{q^{m(m-1)/2}(q^p;q^p)_\infty^4}{(q;q)_\infty^3(q^j,q^{p-j};q^p)_\infty^2}, \tag{4.1.2}$$

and for any integer m and any two coprime integers $2p$ and j such that $p \geq 1$ and $1 \leq j \leq 2p-1$, we have

$$\sum_{n=0}^{\infty} a_{m,j,2p}(2pn+p)q^n = (-1)^m p\frac{q^{m(m-1)/2+(j-1)/2}(-q^j,-q^{p-j};q^p)_\infty(q^{2p};q^{2p})_\infty^4}{(q;q)_\infty^3(q^j,q^{2p-j};q^{2p})_\infty^2(-q^p;q^p)_\infty^2}. \tag{4.1.3}$$

In particular,

$$a_{m,j,p}(pn+pj-j^2) \equiv 0 \pmod{p}$$

and

$$a_{m,j,2p}(2pn+p) \equiv 0 \pmod{p}.$$

Equation (4.1.2) is a mild generalization of (4.1.1), and (4.1.3) is new. We prove (4.1.2) and (4.1.3) in Section 4.3. The method of proof is similar to that of (4.1.1) in [23], but requires more detailed examinations of the products in (4.3.6) and (4.3.8).

Remark: We can also deduce (4.1.2) from (4.1.1) together with the fact that

$$AL(qx, q, z) = 1 - xAL(x, q, z) \qquad \text{[39, Eq. (3.2c)]}.$$

In [31], S. Garthwaite showed that there exists infinitely many congruences for the third order mock theta function, $\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}$. However, explicit congruences for $\omega(q)$ were first given by Waldherr [67]. In the next section, we apply special cases of Theorem 4.1.3 to six mock theta functions. In some of these cases, this yields congruences and partition identities for the mock theta functions. This is the second objective of this chapter.

4.2 Congruences for mock theta functions

In [7], Andrews identified seven functions in Ramanujan's lost notebook [64] as sixth order mock theta functions. Three of them are

$$\psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}},$$
$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}},$$
$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}.$$

For the purpose of this section, we list three other mock theta functions, namely,

$$\psi_-(q) = \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n},$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}},$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}.$$

The function ψ_- is a sixth order mock theta function discovered by B. C. Berndt and S. H. Chan in [15], $B(q)$ is a second order mock theta function discovered by R. McIntosh in [61], and $V_0(q)$ is an eighth order mock theta function discovered by B. Gordon and McIntosh in [37].

Special cases of Theorem 4.1.3 lead to the corollaries below.

Corollary 4.2.1. *Let* $\sum_{n=0}^{\infty} b(n)q^n := \lambda(q)$ *and* $\sum_{n=0}^{\infty} c(n)q^n := \rho(q)$. *Then*

$$\sum_{n=0}^{\infty} b(6n+2)q^n = \sum_{n=0}^{\infty} c(6n+2)q^n \tag{4.2.1}$$

$$= 3\frac{(-q,-q^2;q^3)_\infty(q^6;q^6)_\infty^4}{(q;q)_\infty^3(q,q^5;q^6)_\infty^2(-q^3;q^3)_\infty^2}, \tag{4.2.2}$$

$$\sum_{n=0}^{\infty} b(6n+4)q^n = \sum_{n=0}^{\infty} c(6n+4)q^n \tag{4.2.3}$$

$$= 6\frac{(q^6;q^6)_\infty^4}{(q;q)_\infty^3(q,q^5;q^6)_\infty^2}. \tag{4.2.4}$$

In particular,

$$b(6n+2) = c(6n+2) \equiv 0 \pmod{3},$$

$$b(6n+4) = c(6n+4) \equiv 0 \pmod{6}.$$

Proof. First, we prove (4.2.2) and (4.2.4). By (4.11) in [7],

$$(q,q^5,q^6;q^6)_\infty \rho(q) = \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(3r+4)}}{1-q^{6r+1}}.$$

Dividing both sides by $(q,q^5,q^6;q^6)_\infty$, we find that,

$$q\rho(q) = \sum_{n=0}^{\infty} a_{0,1,6}(n)q^n.$$

Hence, we see that (4.2.2) and (4.2.4) follow from the special cases $(m, j, 2p) = (0, 1, 6)$ in (4.1.3) and $(m, j, p) = (0, 1, 6)$ in (4.1.2), respectively.

Next, we prove (4.2.1) and (4.2.3). We recall $(0.18)_R$ and $(0.21)_R$ in [7],

$$q^{-1}\psi(q^2) + \rho(q) = (-q; q^2)_\infty^2(-q, -q^5, q^6; q^6)_\infty,$$
$$2q^{-1}\psi(q^2) + \lambda(-q) = (-q; q^2)_\infty^2(-q, -q^5, q^6; q^6)_\infty. \tag{4.2.5}$$

Equating the right sides of both equations and subtracting $q^{-1}\psi(q^2)$ from both sides of the resulting equation, we find that

$$\lambda(-q) + q^{-1}\psi(q^2) = \rho(q).$$

Since $q^{-1}\psi(q^2)$ consists entirely of terms of the form q^n where n is odd, we conclude that

$$\sum_{n=0}^{\infty} b(2n)(-q)^{2n} = \sum_{n=0}^{\infty} b(2n)q^{2n} = \sum_{n=0}^{\infty} c(2n)q^{2n}. \tag{4.2.6}$$

Thus (4.2.1) and (4.2.3) follow immediately.

The congruences in the corollary follow immediately from equations (4.2.1)–(4.2.4).

□

Corollary 4.2.2. *Let* $\sum_{n=0}^{\infty} d(n)q^n := \psi_-(q)$ *and* $\sum_{n=0}^{\infty} e(n)q^n := \psi(q)$. *Then*

$$\sum_{n=0}^{\infty} d(3n+2)q^n \equiv \sum_{n=0}^{\infty} e(3n+2)q^n \tag{4.2.7}$$
$$\equiv 2\frac{(q^6; q^6)_\infty}{(q^2, q^3, q^4; q^6)_\infty} \pmod 3. \tag{4.2.8}$$

Proof. From (3.24) in [7],

$$(q; q)_\infty \psi(q) = \frac{2q(q^6; q^6)_\infty^3}{(q^2; q^2)_\infty} - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{(r+1)(3r+2)/2}}{1 - q^{3r+1}}.$$

Dividing both sides by $(q;q)_\infty$, we find that

$$\psi(q) = \frac{2q(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty} - \sum_{n=0}^{\infty} a_{0,1,3}(n)q^n. \tag{4.2.9}$$

Multiplying both the numerator and denominator of the product on the right side by $(q;q)_\infty$, and taking congruence modulo 3, we find that

$$\frac{2q(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{2q(q^6;q^6)_\infty^3}{(q;q)_\infty^2(-q;q)_\infty} \equiv \frac{2q(q^{18};q^{18})_\infty(q;q)_\infty}{(q^3;q^3)_\infty(-q;q)_\infty} \pmod 3. \tag{4.2.10}$$

Next, we recall an identity from [14, p. 49]

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} - 2q(q^3,q^{15},q^{18};q^{18})_\infty. \tag{4.2.11}$$

Since from Theorem 4.1.3, we know that $\sum_{n=0}^{\infty} a_{0,1,3}(3n+2)q^{3n+2} \equiv 0 \pmod 3$, by considering only terms of the form q^n where $n \equiv 2 \pmod 3$ on both sides of (4.2.9), and applying (4.2.10) and (4.2.11), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} e(3n+2)q^{3n+2} &\equiv -4q^2\frac{(q^{18};q^{18})_\infty(q^3,q^{15},q^{18};q^{18})_\infty}{(q^3;q^3)_\infty} \\ &\equiv 2q^2\frac{(q^{18};q^{18})_\infty}{(q^6,q^9,q^{12};q^{18})_\infty} \pmod 3.\end{aligned}$$

Thus we obtain (4.2.8).

By (5.31) in [39],

$$\psi_-(q) = -\frac{1}{2}AL(1,q^3,q) + q\frac{(q^6;q^6)_\infty^3}{2(q;q)_\infty(q^2;q^2)_\infty},$$

we find that,

$$\begin{aligned}\psi_-(q) \equiv 4\psi_-(q) &= -2AL(1,q^3,q) + 2q\frac{(q^6;q^6)_\infty^3}{2(q;q)_\infty(q^2;q^2)_\infty} \\ &\equiv AL(1,q^3,q) + 2q\frac{(q^6;q^6)_\infty^3}{2(q;q)_\infty(q^2;q^2)_\infty} = \psi(q) \pmod 3,\end{aligned}$$

where in the last equality, we invoked (4.2.9). This proves (4.2.7). □

Corollary 4.2.3. *Let* $\sum_{n=0}^{\infty} f(n)q^n := B(q)$. *Then*

$$\sum_{n=0}^{\infty} f(4n+2)q^n = 4\frac{(q^4;q^4)_\infty^4}{(q;q)_\infty^3(q,q^3;q^4)_\infty^2}, \tag{4.2.12}$$
$$\sum_{n=0}^{\infty} f(4n+1)q^n = 2\frac{(-q;q^2)_\infty^2(q^4;q^4)_\infty^4}{(q;q)_\infty^3(q,q^3;q^4)_\infty^2(-q^2;q^2)_\infty^2}. \tag{4.2.13}$$

In particular,

$$f(4n+2) \equiv 0 \pmod 4 \qquad \textit{and} \qquad f(4n+1) \equiv 0 \pmod 2.$$

Proof. By (5.2) in [39],

$$B(q) = -q^{-1}AL(1,q^4,q^3).$$

Multiplying both sides by q, we find that,

$$qB(q) = \sum_{n=0}^{\infty} a_{0,1,4}(n)q^n.$$

Hence, we see that (4.2.12) and (4.2.13) follow from the special case $(m,j,2p) = (0,1,4)$ in (4.1.3) and $(m,j,p) = (0,1,4)$ in (4.1.2), respectively.

□

Corollary 4.2.4. *Let* $\sum_{n=0}^{\infty} g(n)q^n := V_0(q)$. *Then*

$$\sum_{n=0}^{\infty} g(8n+3)q^n = 4\frac{(-q,-q^3;q^4)_\infty(q^8;q^8)_\infty^4}{(q;q)_\infty^3(-q^4;q^4)_\infty^2}\left(\frac{1}{(q,q^7;q^8)_\infty^2} + \frac{q}{(q^3,q^5;q^8)_\infty^2}\right), \tag{4.2.14}$$
$$\sum_{n=0}^{\infty} g(8n+6)q^n = 8\frac{(q^8;q^8)_\infty^4}{(q;q)_\infty^3}\left(\frac{1}{(q,q^7;q^8)_\infty^2} + \frac{q}{(q^3,q^5;q^8)_\infty^2}\right). \tag{4.2.15}$$

In particular,

$$g(8n+3) \equiv 0 \pmod 4 \qquad \textit{and} \qquad g(8n+6) \equiv 0 \pmod 8.$$

Proof. Multiplying both sides of (5.41) in [39] by q, we obtain

$$qV_0(q) = \sum_{n=0}^{\infty} a_{0,1,8}(n)q^n + \sum_{n=0}^{\infty} a_{0,3,8}(n)q^n.$$

Hence, we see that (4.2.14) follows immediately from the special cases $(m, j, 2p) = (0, 1, 8)$ and $(m, j, 2p) = (0, 3, 8)$ in (4.1.3).

Next, we prove (4.2.15). First, we note that

$$\sum_{n=0}^{\infty} a_{0,3,8}(8n+7)q^n = a_{0,3,8}(7)+q\sum_{n=0}^{\infty} a_{0,3,8}(8n+15)q^n = a_{0,3,8}(7)+q\frac{8(q^8;q^8)_\infty^4}{(q;q)_\infty^3(q^3,q^5;q^8)_\infty^2}, \tag{4.2.16}$$

the second equality in (4.2.16) follows from the special case $(m, j, p) = (0, 3, 8)$ in (4.1.2). On the other hand, it's easy to see that $a_{0,3,8}(7) = 0$, so (4.2.15) follows from (4.2.16) and the special case $(m, j, p) = (0, 1, 8)$ in (4.1.2).

□

We end this section with alternative proofs of the congruences for $c(6n + 2)$ and $c(6n + 4)$ in Corollary 4.2.1. We claim that

$$\sum_{n=0}^{\infty} c(2n)q^n \equiv (-q, -q^3, q^4; q^4)_\infty = \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 2, \tag{4.2.17}$$

$$\sum_{n=0}^{\infty} c(2n)q^n \equiv (q^3; q^3)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(3n+1)/2} \pmod 3. \tag{4.2.18}$$

Therefore, $c(2n) \equiv 0 \pmod 2$ whenever n is not a triangular number, and $c(2n) \equiv 0 \pmod 3$ whenever n is not 3 times a pentagonal number. These results imply the congruences for $c(6n + 2)$ and $c(6n + 4)$ in Corollary 4.2.1.

Proof of (4.2.17) *and* (4.2.18). First, note that the equalities in (4.2.17) and (4.2.18) follow from [14, Entry 22(ii) and (iii), p.36]. Next, by using Jacobi's triple product identity [14, pp. 33–36],

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-qz, -q/z, q^2; q^2)_\infty,$$

we know that

$$\begin{aligned}(-q;q^2)_\infty^2(-q,-q^5,q^6;q^6)_\infty &\equiv (q^2;q^4)_\infty(-q,-q^5,q^6;q^6)_\infty \pmod 2\\ &= (q^2;q^4)_\infty\left((-q^8,-q^{16},q^{24};q^{24})_\infty + q(-q^4,-q^{20},q^{24};q^{24})_\infty\right).\end{aligned}$$

Substituting this into the right side of (4.2.5) and extracting terms with even powers of q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c(2n)q^{2n} &\equiv (q^2;q^4)_\infty(-q^8,-q^{16},q^{24};q^{24})_\infty \equiv (q^2;q^4)_\infty(q^8;q^8)_\infty\\ &\equiv (-q^2,-q^6,q^8;q^8)_\infty \pmod 2.\end{aligned}$$

Upon replacing q^2 by q, we complete the proof of (4.2.17).

Finally, we consider

$$\begin{aligned}(-q;q^2)_\infty^2(-q,-q^5,q^6;q^6)_\infty &= \frac{(-q;q^2)_\infty^3(-q,-q^5,q^6;q^6)_\infty}{(-q;q^2)_\infty}\\ &\equiv \frac{(-q^3;q^6)_\infty(-q,-q^5,q^6;q^6)_\infty}{(-q;q^2)_\infty} = (q^6;q^6)_\infty \pmod 3.\end{aligned}$$

Substituting this into the right side of (4.2.5), extracting terms with even powers of q, and replacing q^2 by q, we complete the proof of (4.2.18). □

4.3 Proof of Theorem 4.1.3

We prove (4.1.2) and (4.1.3) in this section. The congruences in Theorem 4.1.3 then follow immediately from (4.1.2) and (4.1.3).

Proof of (4.1.2). Recall from [46, Lemma 3], [21, p. 610]

$$\frac{[a,-b]_\infty(q)_\infty^2}{2[-a,ab]_\infty(-q)_\infty^2}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n b^{-n}q^{n(n+1)/2}}{1+aq^n}+\frac{b[1/b]_\infty}{2(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1-abq^n}. \qquad (4.3.1)$$

Replacing a and b in (4.3.1) by ab and $1/b$, respectively, we obtain

$$\frac{[ab,-1/b]_\infty(q)_\infty^2}{2[-ab,a]_\infty(-q)_\infty^2}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n b^{n}q^{n(n+1)/2}}{1+abq^n}+\frac{[b]_\infty}{2b(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1-aq^n}. \qquad (4.3.2)$$

Replacing q by q^p, setting $(a,b)=(-q^{pm},q^j)$ in (4.3.2), dividing by $q^{-j}(q^j,q^{p-j},q^p;q^p)_\infty$, and noting that

$$[-q^{pm};q^p]_\infty=2q^{-pm(m-1)/2}(-q^p;q^p)_\infty^2,$$

we arrive at

$$\begin{aligned}&\frac{(-1)^m q^{pm(m-1)/2}[-q^j;q^p]_\infty^2(q^p;q^p)_\infty}{4[q^j;q^p]_\infty^2(-q^p;q^p)_\infty^4}\\&=\sum_{n=0}^{\infty}a_{m,j,p}(n)q^n+\frac{1}{2(q^p;q^p)_\infty(-q^p;q^p)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{pn(n+1)/2}}{1+q^{pn+pm}}.\end{aligned} \qquad (4.3.3)$$

We observe that the second series on the right side only contain terms of the form q^n where n is a multiple of p. Therefore it suffices to examine the p-dissection of the product on the left side. Recall that from Ramanujan's ${}_1\psi_1$ summation formula, we have the following corollary

$$\frac{[xy]_\infty(q)_\infty^2}{[x,y]_\infty}=\sum_{n=-\infty}^{\infty}\frac{x^n}{1-yq^n}, \qquad (|q|<|x|<1). \qquad (4.3.4)$$

Replacing q by q^p and substituting $(x,y)=(q^j,-1)$, we find that the formula becomes

$$\frac{[-q^j;q^p]_\infty(q^p;q^p)_\infty^2}{[q^j,-1;q^p]_\infty}=\sum_{n=-\infty}^{\infty}\frac{q^{jn}}{1+q^{pn}}.$$

Separating the series on the right side according to the summation index modulo p, we find that

$$\frac{[-q^j;q^p]_\infty(q^p;q^p)_\infty^2}{[q^j,-1;q^p]_\infty}=\sum_{n=-\infty}^{\infty}\frac{q^{jn}}{1+q^{pn}}$$

$$= \sum_{k\in Z_p} \sum_{n=-\infty}^{\infty} \frac{(q^j)^{pn+k}}{1+q^{p(pn+k)}}$$

$$= \sum_{k\in Z_p} q^{jk} \sum_{n=-\infty}^{\infty} \frac{q^{pjn}}{1+q^{pk+np^2}}$$

$$= \sum_{k\in Z_p} q^{jk} \frac{[-q^{pk+pj}; q^{p^2}]_\infty (q^{p^2}; q^{p^2})_\infty^2}{[q^{pj}, -q^{pk}; q^{p^2}]_\infty}, \tag{4.3.5}$$

where in the last equality, we apply (4.3.4) with q replaced by q^{p^2} and $(x, y) = (q^{pj}, -q^{pk})$ to each of the summand. Here "$\sum_{k\in Z_p}$" means summing over the integer k which runs over a complete set of residues modulo p.

Noting that the left side of (4.3.3) is

$$\frac{(-1)^m q^{pm(m-1)/2}[-q^j; q^p]_\infty^2 (q^p; q^p)_\infty}{4[q^j; q^p]_\infty^2 (-q^p; q^p)_\infty^4} = \frac{(-1)^m q^{pm(m-1)/2}}{(q^p; q^p)_\infty^3} \times \frac{(-q^j, -q^{p-j}, q^p, q^p; q^p)_\infty^2}{(q^j, q^{p-j}, -1, -q^p; q^p)_\infty^2}$$

and substituting (4.3.5) into the left side of (4.3.3), we arrive at

$$\frac{(-1)^m q^{pm(m-1)/2}}{(q^p; q^p)_\infty^3} \left(\sum_{k\in Z_p} q^{jk} \frac{[-q^{pk+pj}; q^{p^2}]_\infty (q^{p^2}; q^{p^2})_\infty^2}{[q^{pj}, -q^{pk}; q^{p^2}]_\infty} \right)^2$$

$$= \sum_{n=0}^{\infty} a_{m,j,p}(n) q^n + \frac{1}{2(q^p; q^p)_\infty (-q^p; q^p)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{q^{pn(n+1)/2}}{1+q^{pn+pm}}. \tag{4.3.6}$$

Next, we extract only terms with q^n where $n \equiv pj - j^2 \pmod{p}$ in (4.3.6). To do that, we see that it suffices to examine the square of the sum of products on the left side. Let us suppose the two sums in the bracket have summation indices k and l and we let k run from 0 to $p-1$. Then for each k, we require l so that $jk + jl \equiv pj - j^2 \pmod{p}$. Since j and p are coprime, this requirement is equivalent to $k + l \equiv p - j \pmod{p}$, thus we may set $l = p - j - k$. Thus, we obtain from (4.3.6),

$$\sum_{n=0}^{\infty} a_{m,j,p}(pn + pj - j^2) q^{pn+pj-j^2}$$

$$= \frac{(-1)^m q^{pm(m-1)/2}}{(q^p; q^p)_\infty^3} \sum_{k=0}^{p-1} q^{jk} \frac{[-q^{pk+pj}; q^{p^2}]_\infty (q^{p^2}; q^{p^2})_\infty^2}{[q^{pj}, -q^{pk}; q^{p^2}]_\infty}$$

$$\times\, q^{j(p-j-k)} \frac{[-q^{p(p-k)}; q^{p^2}]_\infty (q^{p^2}; q^{p^2})_\infty^2}{[q^{pj}, -q^{p(p-j-k)}; q^{p^2}]_\infty}$$

$$= \frac{(-1)^m q^{pm(m-1)/2}}{(q^p; q^p)_\infty^3} \sum_{k=0}^{p-1} q^{j(p-j)} \frac{(q^{p^2}; q^{p^2})_\infty^4}{[q^{pj}; q^{p^2}]_\infty^2}$$

$$= (-1)^m p q^{j(p-j)+pm(m-1)/2} \frac{(q^{p^2}; q^{p^2})_\infty^4}{(q^p; q^p)_\infty^3 (q^{pj}, q^{p^2-pj}; q^{p^2})_\infty^2}. \tag{4.3.7}$$

Replacing q^p by q, we obtain (4.1.2). □

Proof of (4.1.3). Replacing p by $2p$ in (4.3.6), we obtain

$$\frac{(-1)^m q^{pm(m-1)}}{(q^{2p}; q^{2p})_\infty^3} \left(\sum_{k \in Z_{2p}} \frac{q^{jk}[-q^{2pk+2pj}; q^{4p^2}]_\infty (q^{4p^2}; q^{4p^2})_\infty^2}{[q^{2pj}, -q^{2pk}; q^{4p^2}]_\infty} \right)^2$$

$$= \sum_{n=0}^{\infty} a_{m,j,2p}(n) q^n + \frac{1}{2(q^{2p}; q^{2p})_\infty (-q^{2p}; q^{2p})_\infty^2} \sum_{n=-\infty}^{\infty} \frac{q^{pn(n+1)}}{1 + q^{2pn+2pm}}. \tag{4.3.8}$$

Next, we extract only terms with q^n where $n \equiv p \pmod{2p}$ in (4.3.8) using a similar argument as in the proof of (4.1.2). Suppose the two sums in the bracket have summation indices k and l and we let k run from 0 to $2p-1$. Then for each k, we require l so that $jk + jl \equiv p \pmod{2p}$. Since j and $2p$ are coprime (and hence j must be odd), this requirement is equivalent to $k + l \equiv p \equiv 3p \pmod{2p}$. Hence we my set $l = p - k$ for $0 \le k \le p-1$, and $l = 3p - k$ for $p \le k \le 2p-1$. Thus, we obtain from (4.3.8),

$$\sum_{n=0}^{\infty} a_{m,j,2p}(2pn+p) q^{2pn+p} = \frac{(-1)^m q^{pm(m-1)}}{(q^{2p}; q^{2p})_\infty^3} \sum_{k=0}^{p-1} \frac{q^{jk}[-q^{2pk+2pj}; q^{4p^2}]_\infty (q^{4p^2}; q^{4p^2})_\infty^2}{[q^{2pj}, -q^{2pk}; q^{4p^2}]_\infty}$$

$$\times\, q^{j(p-k)} \frac{[-q^{2p(p-k)+2pj}; q^{4p^2}]_\infty (q^{4p^2}; q^{4p^2})_\infty^2}{[q^{2pj}, -q^{2p(p-k)}; q^{4p^2}]_\infty}$$

$$+ \frac{(-1)^m q^{pm(m-1)}}{(q^{2p}; q^{2p})_\infty^3} \sum_{k=p}^{2p-1} \frac{q^{jk}[-q^{2pk+2pj}; q^{4p^2}]_\infty (q^{4p^2}; q^{4p^2})_\infty^2}{[q^{2pj}, -q^{2pk}; q^{4p^2}]_\infty}$$

$$\times\, q^{j(3p-k)} \frac{[-q^{2p(3p-k)+2pj}; q^{4p^2}]_\infty (q^{4p^2}; q^{4p^2})_\infty^2}{[q^{2pj}, -q^{2p(3p-k)}; q^{4p^2}]_\infty}. \tag{4.3.9}$$

Replacing k by $p+k$ in the second sum on the right side of (4.3.9) and simplifying, we find

$$\sum_{n=0}^{\infty} a_{m,j,2p}(2pn+p)q^{2pn+p} = \frac{(-1)^m q^{pm(m-1)+jp}(q^{4p^2};q^{4p^2})_\infty^4}{[q^{2pj};q^{4p^2}]_\infty^2 (q^{2p};q^{2p})_\infty^3}$$
$$\times \sum_{k=0}^{p-1} \frac{[-q^{2pk+2pj}, -q^{2p(p-k)+2pj}; q^{4p^2}]_\infty}{[-q^{2pk}, -q^{2p(p-k)}; q^{4p^2}]_\infty} + \frac{(-1)^m q^{pm(m-1)+jp}(q^{4p^2};q^{4p^2})_\infty^4}{[q^{2pj};q^{4p^2}]_\infty^2 (q^{2p};q^{2p})_\infty^3}$$
$$\times \sum_{k=0}^{p-1} q^{2pj} \frac{[-q^{2p^2+2pk+2pj}, -q^{4p^2-2pk+2pj}; q^{4p^2}]_\infty}{[-q^{2p^2+2pk}, -q^{4p^2-2pk}; q^{4p^2}]_\infty}. \tag{4.3.10}$$

Dividing (4.3.10) throughout by q^p and then replacing q^{2p} by q, we arrive at

$$\sum_{n=0}^{\infty} a_{m,j,2p}(2pn+p)q^n = \frac{(-1)^m q^{m(m-1)/2+(j-1)/2}(q^{2p};q^{2p})_\infty^4}{[q^j;q^{2p}]_\infty^2 (q;q)_\infty^3} \sum_{k=0}^{p-1} \frac{[-q^{k+j}, -q^{p-k+j}; q^{2p}]_\infty}{[-q^k, -q^{p-k}; q^{2p}]_\infty}$$
$$+ \frac{(-1)^m q^{m(m-1)/2+(j-1)/2}(q^{2p};q^{2p})_\infty^4}{[q^j;q^{2p}]_\infty^2 (q;q)_\infty^3}$$
$$\times \sum_{k=0}^{p-1} q^j \frac{[-q^{p+k+j}, -q^{2p-k+j}; q^{2p}]_\infty}{[-q^{p+k}, -q^{2p-k}; q^{2p}]_\infty}.$$

Comparing with (4.1.3), we see that it remains to show that

$$2p\frac{[-q^j;q^p]_\infty}{[-1;q^p]_\infty} = \sum_{k=0}^{p-1}\left(\frac{[-q^{k+j}, -q^{p-k+j}; q^{2p}]_\infty}{[-q^k;q^p]_\infty} + q^j \frac{[-q^{p-k-j}, -q^{k-j}; q^{2p}]_\infty}{[-q^k;q^p]_\infty} \right). \tag{4.3.11}$$

With q replaced by q^{2p}, setting $(A,b,c,d,e) = (-q^p, -q^j, -q^k, -q^{p-k}, -q^{p-j}]$ and $(A,b,c,d,e) = (-q^{p+k}, -1, -q^{k+j}, -q^{p+k-j}, -q^p)$, respectively, in [28, Eq. (3.1)],

$$[A/b, A/c, A/d, A/e; q]_\infty - [b,c,d,e;q]_\infty = b[A, A/bc, A/bd, A/be; q]_\infty,$$

we find that

$$[q^{p-j}, q^{p-k}, q^k, q^j; q^{2p}]_\infty - [-q^j, -q^k, -q^{p-k}, -q^{p-j}; q^{2p}]_\infty$$
$$= -q^j[-q^p, -q^{p-k-j}, -q^{k-j}, -1; q^{2p}]_\infty \tag{4.3.12}$$

and

$$[q^{p+k}, q^{p-j}, q^j, q^k; q^{2p}]_\infty - [-1, -q^{k+j}, -q^{p-k+j}, -q^p; q^{2p}]_\infty$$
$$= -[-q^{p+k}, -q^{p-j}, -q^j, -q^k; q^{2p}]_\infty. \tag{4.3.13}$$

Subtracting (4.3.13) from (4.3.12), noting that $[x; q^{2p}]_\infty = [q^{2p}/x; q^{2p}]_\infty$ and rearranging, we arrive at

$$[-1, -q^{k+j}, -q^{p-k+j}, -q^p; q^{2p}]_\infty + q^j[-q^p, -q^{p-k-j}, -q^{k-j}, -1; q^{2p}]_\infty$$
$$= 2[-q^j, -q^k, -q^{p-k}, -q^{p-j}; q^{2p}]_\infty. \tag{4.3.14}$$

Dividing both sides of (4.3.14) by $[-1, -q^k; q^p]_\infty$, we find

$$\frac{[-q^{k+j}, -q^{p-k+j}; q^{2p}]_\infty}{[-q^k; q^p]_\infty} + q^j \frac{[-q^{p-k-j}, -q^{k-j}; q^{2p}]_\infty}{[-q^k; q^p]_\infty} = 2\frac{[-q^j; q^p]_\infty}{[-1; q^p]_\infty},$$

which implies (4.3.11), and this completes our proof of (4.1.3). □

Chapter 5

Pairs of partitions without repeated odd parts

5.1 Introduction

A partition π of n is a sequence of nonincreasing integers, the sum of which equals n, while an overpartition λ of n is a partition of n where the first occurrence of a number may be overlined. In [18], Bringmann and Lovejoy considered *overpartition pairs* (λ, μ) of n, which is a pair of overpartitions, with the sum of all parts equaling n. Define $\overline{pp}(n)$ as the number of overpartition pairs of n. Then from [18], the function $\overline{pp}(n)$ has the generating function

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}.$$

A rank statistic for $\overline{pp}(n)$ is given in [18]. By giving an appropriate ordering to the parts of an overpartition pair (λ, μ), Bringmann and Lovejoy defined the rank of an overpartition pair (λ,μ) as

$$\ell(\lambda,\mu)-n(\lambda)-\overline{n}(\mu)-\chi(\lambda,\mu),$$

where $\ell(\lambda,\mu)$, $n(\lambda)$, and $\overline{n}(\mu)$ denote the largest part of (λ,μ), largest part of λ, and largest overlined part of μ, respectively, and $\chi(\lambda,\mu)$ is defined to be 1 if the largest part of (λ,μ) is non-overlined in μ and 0 otherwise. Define also $\overline{NN}(m,n)$ as the number of overpartition pairs of n whose rank is m, and $\overline{NN}(t,l,n)$ as the number of overpartition pairs of n whose rank is congruent to t modulo l. It is shown in [18, Proposition 2.1] that $\overline{NN}$ has the generating function

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{NN}(m,n)z^m q^n=\sum_{n=0}^{\infty}\frac{(-1;q)_n^2 q^n}{(zq,q/z;q)_n},$$

and from [18, Theorem 1.2]

$$\overline{NN}(r,3,3n+2)=\frac{\overline{pp}(3n+2)}{3}, \tag{5.1.1}$$

which gives a combinatorial interpretation of the congruence [18, Corollary 1.3]

$$\overline{pp}(3n+2)\equiv 0 \pmod 3. \tag{5.1.2}$$

Results on rank differences modulo 3 and 4 are given in [18]; in particular the generating function for the rank difference modulo 3 [18, Theorem 1.4] is,

$$\begin{aligned}4+&\sum_{n=1}^{\infty}\left(\overline{NN}(0,3,n)-\overline{NN}(1,3,n)\right)q^n\\ &=4\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty}+4q\frac{(q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty(q^9;q^9)_\infty}.\end{aligned}$$

The proof of (5.1.1) and results on rank differences depend on the identity [53, eq. (1.11)],

$$\frac{4}{(1+z)(1+z^{-1})}+\sum_{n=1}^{\infty}\frac{(-1;q)_n^2 q^n}{(zq,q/z;q)_n}=\frac{4(-q;q)_\infty^2}{(1+z)(1+z^{-1})(zq,q/z;q)_\infty}, \tag{5.1.3}$$

which relates the generating function for ranks of overpartition pairs to a generating function for the *cranks of overpartition pairs* $\frac{(-q;q)^2_\infty}{(zq,q/z;q)_\infty}$. The result (5.1.1) is an analog of the combinatorial interpretation of two of the Ramanujan congruences for the partition function $p(n)$,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7}.$$

Define

$$\sum_{n=0}^{\infty} t(n)q^n := \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Then $t(n)$ counts the number of partitions of n without repeated odd parts. In [56], Lovejoy and Osburn evaluated rank differences for $t(n)$. We shall count partitions pairs (λ, μ) of n, where each partition, λ and μ, does not have repeated odd parts, and the sum of all the parts of λ and μ is n. We shall simply call them *partition pairs of n without repeated odd parts.* Let $tt(n)$ denote the number of such partition pairs of n. It is clear that $tt(n)$ has the generating function

$$\sum_{n=0}^{\infty} tt(n)q^n = \frac{(-q;q^2)^2_\infty}{(q^2;q^2)^2_\infty}.$$

We state an analogous result to (5.1.2), which first appeared in W.Y.C. Chen and B.L.S. Lin [27, Corollary 2.1].

Theorem 5.1.1. *We have* $tt(3n+2) \equiv 0 \pmod{3}$.

We remark that from [22, Thm 1.6 and Sect. 3], we see that the function $tt(n)$ has congruences

$$tt(25n+14) \equiv tt(25n+24) \equiv 0 \pmod{5}.$$

Recently, P. C. Toh [66] gave an extensive discussion on pairs of various partition functions satisfying congruences modulo 3.

The first objective of this chapter is to present the following two identities, the first of which gives a generalization of (5.1.3). We prove these two identities in Section 3.

Theorem 5.1.2. *We have*

$$\sum_{n=0}^{\infty}\frac{(x,1/x;q)_n q^n}{(zq,q/z;q)_n}=\frac{(1-z)^2}{(1-z/x)(1-xz)}+\frac{z(x,1/x;q)_\infty}{(1-z/x)(1-xz)(zq,q/z;q)_\infty},\tag{5.1.4}$$

$$\sum_{n=0}^{\infty}\frac{(x,q/x;q)_n q^n}{(z,q/z;q)_{n+1}}=\frac{1}{x(1-z/x)(1-q/(xz))}+\frac{[x;q]_\infty}{z(1-x/z)(1-q/(xz))[z;q]_\infty}.\tag{5.1.5}$$

In [54], Lovejoy and Olivier Mallet defined two very general generating functions for overpartition pairs, and showed how one can obtain many q-series identities related to overpartitions from their result. We remark that (5.1.4) and (5.1.5) do not follow from results in that paper.

Combining the summand for $n=0$ on the left side of (5.1.4) with the first product on the right side, we obtain

$$z\frac{(1-x)(1-1/x)}{(1-z/x)(1-xz)}+\sum_{n=1}^{\infty}\frac{(x,1/x;q)_n q^n}{(qz,q/z;q)_n}=\frac{z(x,1/x;q)_\infty}{(1-z/x)(1-xz)(zq,q/z;q)_\infty}.$$

From this identity, setting $x=-1$ we recover (5.1.3). Similarly, dividing both sides by $(1-x)(1-1/x)$ and setting $x=1$, we recover another q-identity of Lovejoy, which we state in the following corollary.

Corollary 5.1.3. *(Lovejoy [53, Thm 1.4]) We have*

$$\frac{4z}{(1+z)^2}+\sum_{n=1}^{\infty}\frac{(-1;q)_n^2 q^n}{(zq,q/z;q)_n}=\frac{4z(q^2;q^2)_\infty}{(1+z)(q;q^2)_\infty(q,zq,q/z;q)_\infty},$$

$$\frac{z}{(1-z)^2}+\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}^2q^n}{(zq,q/z;q)_n}=\frac{z(q;q)_\infty^2}{(1-z)^2(zq,q/z;q)_\infty}.$$

Similarly, we consider combining the summand for $n=0$ on the left side of (5.1.5) and the first product on the right side. Setting $x=-1$ and dividing both sides by $(x-1)$ and setting $x=1$ gives the following two identities, respectively.

Corollary 5.1.4. *We have*

$$\frac{2}{(1-z^2)(1-q^2/z^2)}+\sum_{n=1}^{\infty}\frac{(-1,-q;q)_nq^n}{(z,q/z;q)_{n+1}}=\frac{2(-q;q)_\infty^2}{(1+z)(1+q/z)[z;q]_\infty},$$
$$\frac{(1-q)}{(1-z)^2(1-q/z)^2}-\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}(q;q)_nq^n}{(z,q/z;q)_{n+1}}=\frac{(q;q)_\infty^2}{(1-z)(1-q/z)[z;q]_\infty}.$$

We could also make different specializations to (5.1.4) and (5.1.5) to get results analogous to Corollary 5.1.3.

Corollary 5.1.5. *We have*

$$\frac{(1+qz)(1+q/z)}{(1+q)}\sum_{n=0}^{\infty}\frac{(-q,-1/q;q^2)_nq^{2n}}{(q^2z,q^2/z;q^2)_n}=\frac{q(1-z)^2}{z(1+q)}+\frac{(-q;q^2)_\infty^2}{(zq^2,q^2/z;q^2)_\infty},\tag{5.1.6}$$
$$\frac{z(1+q/z)^2}{(1-q^2/z)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n^2q^{2n}}{(zq^2,q^4/z;q^2)_n}=z-1+\frac{(-q;q^2)_\infty^2}{(zq^2,q^2/z;q^2)_\infty}.\tag{5.1.7}$$

Identity (5.1.6) is obtained by replacing q by q^2 and setting $x=-q$ in (5.1.4), multiplying both sides of the resultant identity by $\frac{(1+zq)(1+q/z)}{(1+q)}$ and simplifying. Identity (5.1.7) is obtained by replacing q by q^2 and setting $x=-q$ in (5.1.5), multiplying both sides of the resultant identity by $z(1+q/z)^2(1-z)$ and simplifying.

Setting $z=1$ in (5.1.6) and (5.1.7), respectively, we recover on the right sides, the generating function for $tt(n)$, $\frac{(-q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}$. These suggest defining rank type functions

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}TT_1(m,n)z^mq^n:=\frac{(1+qz)(1+q/z)}{(1+q)}\sum_{n=0}^{\infty}\frac{(-q,-1/q;q^2)_nq^{2n}}{(q^2z,q^2/z;q^2)_n}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} TT_2(m,n)z^m q^n := \frac{z(1+q/z)^2}{(1-q^2/z)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n^2 q^{2n}}{(zq^2,q^4/z;q^2)_n}.$$

It is obvious from the definition that $TT_1(m,n) = TT_1(-m,n)$ while from (5.1.7), it is obvious that $TT_2(m,n) = TT_2(-m,n)$ except for $(m,n) = (\pm 1, 0)$.

To describe what $TT_1(m,n)$ and $TT_2(m,n)$ count, we first define the *rank* of a partition pair without repeated odd parts. We order the parts in a partition pair (λ,μ) by stipulating that for a number k,

$$k_\lambda > k_\mu,$$

where the subscript indicates to which of the two partitions the part belongs. Similar to [18], we use the notations $l(\cdot)$, $n(\cdot)$, and $n_o(\cdot)$ for the largest part, the number of parts, and the number of odd parts of an object, respectively.

Definition 5.1.6. *The rank of a partition pair without repeated odd parts, (λ,μ), is*

$$\left\lceil \frac{l(\lambda,\mu)}{2} \right\rceil - n(\lambda) - n_o(\mu),$$

where $\lceil\cdot\rceil$ is the ceiling function.

Theorem 5.1.7. *Let $TT(0,0) = 1$, and for $m \neq 0$, let $TT(m,0) = 0$. For $n \geq 1$, let $TT(m,n)$ denote the number of partition pairs of n without repeated odd parts with rank m. Then we have the following generating function,*

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} TT(m,n)z^m q^n = \frac{(-q;q^2)_\infty^2}{(zq^2,q^2/z;q^2)_\infty}.$$

We accomplish the following in Section 2. We give a proof of Theorem 5.1.7. Next, we apply Theorem 5.1.7 to give descriptions of $TT_1(m,n)$ and $TT_2(m,n)$. Finally, we show that all three partition statistics $TT(m,n)$, $TT_1(m,n)$, and $TT_2(m,n)$ give combinatorial explanations of the congruence in Theorem 5.1.1.

Remark: By examining the product representation in Theorem 5.1.7, we define the crank of a partition pair without repeated odd parts, (λ, μ), as the number of even parts in λ minus the number of even parts in μ. This way, both the rank and the crank for partition pairs without repeated odd parts have the same generating function. Unlike the ranks and cranks for the ordinary partition, in this case, the number of such partition pairs (λ, μ) of n with crank m equals to the number of the partition pairs with rank m. It would be very interesting if one can find a bijection for this.

5.2 Ranks

We first give a proof of Theorem 5.1.7. Our proof follows the method of proof of Proposition 2.1 in [18].

Proof of Theorem 5.1.7. We split the partition pairs without repeated odd parts into four cases, depending on whether the largest part is even or odd and whether it is in λ or μ, then we get four series. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-q/z;q^2)_n^2 q^{2n} z^{n-1}}{(q^2/z,q^2;q^2)_n}$$

is the generating function for partition pairs without repeated odd parts whose largest part $2n$ is in λ, where the exponent of q is the number being partitioned and the exponent of z is the rank. Combining this with the three other cases gives

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} TT(m,n) z^m q^n$$

$$
\begin{aligned}
&= 1 + \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_n^2 q^{2n} z^{n-1}}{(q^2/z,q^2;q^2)_n} + \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_{n-1}^2 q^{2n-1} z^{n-1}}{(q^2/z,q^2;q^2)_{n-1}} \\
&\quad + \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_n^2 q^{2n} z^{n}}{(q^2/z;q^2)_{n-1}(q^2;q^2)_n} + \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_{n-1}(-q/z;q^2)_n q^{2n-1} z^{n-1}}{(q^2/z,q^2;q^2)_{n-1}} \\
&= 1 + q + (1+qz) \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_n^2 q^{2n} z^{n-1}}{(q^2/z,q^2;q^2)_n} \\
&\quad + (1+qz) \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_{n-1}(-q/z;q^2)_n q^{2n-1} z^{n-1}}{(q^2/z;q^2)_{n-1}(q^2;q^2)_n} \\
&= 1 + q + (1+qz) \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_{n-1}(-q/z;q^2)_n q^{2n-1} z^{n-1}}{(q^2/z;q^2)_{n-1}(q^2;q^2)_n} \times \left(1 + q\frac{1+q^{2n-1}/z}{1-q^{2n}/z}\right) \\
&= (1+q)\left(1 + (1+qz) \sum_{n=1}^{\infty} \frac{(-q/z;q^2)_{n-1}(-q/z;q^2)_n q^{2n-1} z^{n-1}}{(q^2/z;q^2)_n(q^2;q^2)_n}\right) \\
&= (1+q) \sum_{n=0}^{\infty} \frac{(\frac{-1}{qz};q^2)_n(-q/z;q^2)_n q^{2n} z^{n}}{(q^2/z;q^2)_n(q^2;q^2)_n} \\
&= \frac{(-q;q^2)_\infty^2}{(zq^2,q^2/z;q^2)_\infty},
\end{aligned}
$$

where in last equality, we invoked the q-Gauss summation [37],

$$
\sum_{n\geq 0} \frac{(a,b)_n (c/ab)^n}{(c,q)_n} = \frac{(c/a,c/b)_\infty}{(c,c/ab)_\infty}.
$$

□

By (5.1.6) and (5.1.7), we have the following corollary.

Corollary 5.2.1. *For $n \geq 1$, we have*

$$
TT_1(m,n) = \begin{cases} TT(m,n) - (-1)^n, & \text{for } m = \pm 1; \\ TT(0,n) + 2(-1)^n, & \text{for } m = 0; \\ TT(m,n), & \text{otherwise.} \end{cases}
$$

For $n \geq 1$, $TT_2(m,n) = TT(m,n)$ and counts the number of partition pairs of n without repeated odd parts with rank m.

Let $TT(t,l,n) = \sum_{m\equiv t \pmod{l}} TT(m,n)$, and define $TT_1(t,l,n)$ and $TT_2(t,l,n)$ similarly. Setting $z = \omega$, a third root of unity, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{r=0}^{2} TT(r,3,n)\omega^r q^n &= \frac{(-q;q^2)_\infty^2}{(\omega q^2, q^2/\omega; q^2)_\infty} = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty}{(q^6;q^6)_\infty} = \frac{1}{(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty} q^{n^2} \\ &= \frac{1}{(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{9n^2} + \sum_{n=-\infty}^{\infty} q^{9n^2+6n+1} + \sum_{n=-\infty}^{\infty} q^{9n^2-6n+1}\right) \\ &= \frac{(-q^9,-q^9,q^{18};q^{18})_\infty}{(q^6;q^6)_\infty} + 2q\frac{(-q^3,-q^{15},q^{18};q^{18})_\infty}{(q^6;q^6)_\infty},\end{aligned} \tag{5.2.1}$$

where in the second equality, we applied the Jacobi triple product identity [14, pp. 33–36],

$$\sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{5.2.2}$$

with $a = b = q$ and in the last equality, we applied (5.2.2) thrice, with $a = b = -q^9$, $a = -q^3, b = -q^{15}$, and $a = -q^{15}, b = -q^3$, respectively.

For any natural number n, by examining the terms q^{3n+2} in (5.2.1), we find that for $n \geq 0$,

$$TT(r,3,3n+2) = \frac{tt(3n+2)}{3} \qquad \text{for } 0 \leq r \leq 2.$$

By Corollary 5.2.1, we also have

$$\begin{aligned} TT_1(0,3,3n+2) &= \frac{tt(3n+2)}{3} + 2(-1)^n, \\ TT_1(1,3,3n+2) = TT_1(2,3,3n+2) &= \frac{tt(3n+2)}{3} - (-1)^n, \\ TT_2(r,3,3n+2) &= \frac{tt(3n+2)}{3} \qquad \text{for } 0 \leq r \leq 2. \end{aligned}$$

Each of these gives a combinatorial interpretation of the result $tt(3n+2) \equiv 0 \pmod{3}$ stated in Theorem 5.1.1.

5.3 Proof of Theorem 5.1.2

We first prove (5.1.4). We require the following lemma in our proof.

Lemma 5.3.1. *We have the identity,*

$$\frac{(q;q)_\infty^2}{[x;q]_\infty}\frac{xz}{(1-x)(1-z/x)(1-xz)}+\frac{(q;q)_\infty^2}{[z;q]_\infty}\frac{zx}{(1-z)(1-x/z)(1-xz)}$$
$$=\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^{(k^2+5k)/2}}{(1-xq^k)(1-q^k/x)(1-q^k/z)(1-zq^k)}. \tag{5.3.1}$$

Proof. Setting $r=3, s=4$ in [21, (2.2)], we have

$$\frac{(a_1q,q/a_1,a_2q,q/a_2,q/a_3q,q;q)_\infty}{[b_1,b_2,b_3,b_4;q]_\infty}$$
$$=\frac{[a_1/b_1,a_2/b_1,a_3/b_1;q]_\infty}{[b_2/b_1,b_3/b_1,b_4/b_1;q]_\infty}$$
$$\times\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_1^3a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_1q^k)(1-b_1q^k/a_1)(1-b_1q^k/a_2)(1-b_1q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_2b_3b_4}\right)^k$$
$$+\frac{[a_1/b_2,a_2/b_2,a_3/b_2;q]_\infty}{[b_1/b_2,b_3/b_2,b_4/b_2;q]_\infty}$$
$$\times\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_2^3a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_2q^k)(1-b_2q^k/a_1)(1-b_2q^k/a_2)(1-b_2q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_1b_3b_4}\right)^k$$
$$+\frac{[a_1/b_3,a_2/b_3,a_3/b_3;q]_\infty}{[b_1/b_3,b_2/b_3,b_4/b_3;q]_\infty}$$
$$\times\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_3^3a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_3q^k)(1-b_3q^k/a_1)(1-b_3q^k/a_2)(1-b_3q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_1b_2b_4}\right)^k$$
$$+\frac{[a_1/b_4,a_2/b_4,a_3/b_4;q]_\infty}{[b_1/b_4,b_2/b_4,b_3/b_4;q]_\infty}$$
$$\times\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+k)/2}b_4^3a_1^{-1}a_2^{-1}a_3^{-1}}{(1-b_4q^k)(1-b_4q^k/a_1)(1-b_4q^k/a_2)(1-b_4q^k/a_3)}\left(\frac{a_1a_2a_3q^3}{b_1b_2b_3}\right)^k.$$

Letting $a_1\to b_1$, $a_2\to b_2$, $a_3\to b_3$, we obtain

$$\frac{(q;q)_\infty^2}{(1-b_1)(1-b_2)(1-b_3)[b_4;q]_\infty}$$

$$
\begin{aligned}
&= \frac{(q;q)_\infty^2}{[b_4/b_1;q]_\infty} \frac{b_1^2 b_2^{-1} b_3^{-1}}{(1-b_1)(1-b_1/b_2)(1-b_1/b_3)} \\
&\quad + \frac{(q;q)_\infty^2}{[b_4/b_2;q]_\infty} \frac{b_2^2 b_1^{-1} b_3^{-1}}{(1-b_2)(1-b_2/b_1)(1-b_2/b_3)} \\
&\quad + \frac{(q;q)_\infty^2}{[b_4/b_3;q]_\infty} \frac{b_3^2 b_1^{-1} b_2^{-1}}{(1-b_3)(1-b_3/b_1)(1-b_3/b_2)} \\
&\quad + \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1} q^{(k^2+7k)/2} b_4^3 b_1^{-1} b_2^{-1} b_3^{-1}}{(1-b_4q^k)(1-b_4q^k/b_1)(1-b_4q^k/b_2)(1-b_4q^k/b_3)}.
\end{aligned}
\tag{5.3.2}
$$

Setting $b_1 = x^2$, $b_2 = xz$, $b_3 = x/z$, $b_4 = xq$, we arrive at

$$
\begin{aligned}
&\frac{(q;q)_\infty^2}{(1-x^2)(1-xz)(1-x/z)[xq;q]_\infty} \\
&\quad = \frac{(q;q)_\infty^2}{[q/x;q]_\infty} \frac{x^2}{(1-x^2)(1-x/z)(1-xz)} \\
&\qquad + \frac{(q;q)_\infty^2}{[q/z;q]_\infty} \frac{z^3/x}{(1-xz)(1-z/x)(1-z^2)} \\
&\qquad + \frac{(q;q)_\infty^2}{[zq;q]_\infty} \frac{z^{-3}x^{-1}}{(1-x/z)(1-1/(xz))(1-1/z^2)} \\
&\qquad + \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1} q^{(k^2+7k+6)/2}/x}{(1-xq^{k+1})(1-q^{k+1}/x)(1-q^{k+1}/z)(1-zq^{k+1})}.
\end{aligned}
$$

Replacing k by $k-1$ in the series on the right side, multiplying both sides by x and simplifying, we find that

$$
\begin{aligned}
&-\frac{x^2(q;q)_\infty^2}{(1-x^2)(1-xz)(1-x/z)[x;q]_\infty} \\
&\quad = \frac{(q;q)_\infty^2}{[x;q]_\infty} \frac{x^3}{(1-x^2)(1-x/z)(1-xz)} \\
&\qquad + \frac{(q;q)_\infty^2}{[z;q]_\infty} \frac{z^3}{(1-xz)(1-z/x)(1-z^2)} \\
&\qquad - \frac{(q;q)_\infty^2}{[z;q]_\infty} \frac{z^{-2}}{(1-x/z)(1-1/(xz))(1-1/z^2)} \\
&\qquad + \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(k^2+5k)/2}}{(1-xq^k)(1-q^k/x)(1-q^k/z)(1-zq^k)}.
\end{aligned}
$$

Combining the product on the left side with the first product on the right side, combining the second and third products on the right side, and rearranging, we have

$$\frac{(q;q)_\infty^2}{[x;q]_\infty}\frac{xz}{(1-x)(1-z/x)(1-xz)}+\frac{(q;q)_\infty^2}{[z;q]_\infty}\frac{zx}{(1-z)(1-x/z)(1-xz)}$$
$$=\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^{(k^2+5k)/2}}{(1-xq^k)(1-q^k/x)(1-q^k/z)(1-zq^k)},$$

and this completes the proof.

□

Proof of Identity (5.1.4). From a limiting case of Watson's ${}_8\phi_7$ transformation, [14, Eq. (7.2), p. 16]

$$\sum_{n=0}^{\infty}\frac{(aq/bc,d,e;q)_n(\frac{aq}{de})^n}{(q,aq/b,aq/c;q)_n}$$
$$=\frac{(aq/d,aq/e;q)_\infty}{(aq,aq/de;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d,e;q)_n(1-aq^{2n})(-a^2)^n q^{n(n+3)/2}}{(q,aq/b,aq/c,aq/d,aq/e;q)_n(1-a)(bcde)^n}, \tag{5.3.3}$$

we set $a=1$, $b=1/z$, $c=z$, $d=x$, $e=1/x$. Upon simplifying, we find that

$$\sum_{n=0}^{\infty}\frac{(x,1/x;q)_n q^n}{(qz,q/z;q)_n}$$
$$=\frac{(xq,q/x;q)_\infty}{(q;q)_\infty^2}\left(1+\sum_{n=1}^{\infty}\frac{(1+q^n)(1-1/z)(1-z)(1-1/x)(1-x)(-1)^n q^{(n^2+3n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)}\right). \tag{5.3.4}$$

Noting that

$$\sum_{n=1}^{\infty}\frac{(-1)^n q^{(n^2+3n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)}$$
$$=\sum_{n=-\infty}^{-1}\frac{(-1)^n q^{(n^2+5n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)},$$

we have

$$1+\sum_{n=1}^{\infty}\frac{(1+q^n)(1-1/z)(1-z)(1-1/x)(1-x)(-1)^n q^{(n^2+3n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)}$$

$$= (1-1/z)(1-z)(1-1/x)(1-x)\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n^2+5n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)}, \tag{5.3.5}$$

since the series is absolutely convergent for $|q| < 1$. Substituting (5.3.5) into the series in the parenthesis on the right side of (5.3.4), we find that

$$\sum_{n=0}^{\infty}\frac{(x,1/x;q)_n q^n}{(zq,q/z;q)_n} = \frac{(xq,q/x;q)_\infty}{(q;q)_\infty^2}(1-1/z)(1-z)(1-1/x)(1-x)$$
$$\times \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n^2+5n)/2}}{(1-zq^n)(1-q^n/z)(1-q^n/x)(1-xq^n)}.$$

Finally, invoking (5.3.1) on the right side, we arrive at

$$\sum_{n=0}^{\infty}\frac{(x,1/x;q)_n q^n}{(qz,q/z;q)_n} = \frac{(1-z)^2}{(1-z/x)(1-xz)} + \frac{z(x,1/x;q)_\infty}{(1-z/x)(1-xz)(zq,q/z;q)_\infty},$$

and this completes the proof of (5.1.4). □

Next, we prove (5.1.5). As our method of proof is similar to that of (5.1.4), we are more brief in our proof. We require the following lemma.

Lemma 5.3.2. *We have*

$$\frac{(q;q)_\infty^2}{z(1-x/z)(1-q/(xz))[z;q]_\infty} - \frac{(q;q)_\infty^2}{z(1-x/z)(1-q^2/(xz))[x;q]_\infty}$$
$$= \sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}q^{(k^2+7k+2)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)}. \tag{5.3.6}$$

Identity (5.3.6) is obtained by substituting $b_1 = q/z^2, b_2 = x/z, b_3 = q/(xz), b_4 = q/z$ in (5.3.2) and applying elementary manipulations. The proof is straightforward and so we omit the details.

Proof of (5.1.5). Setting $a = q$, $b = z$, $c = q/z$, $d = x$, $e = q/x$, in (5.3.3), we obtain

$$\sum_{n=0}^{\infty}\frac{(x,q/x;q)_n q^n}{(zq,q^2/z;q)_n} = \frac{(q^2/x,xq;q)_\infty}{(q^2,q;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-q^{2n+1})(z,q/z,x,q/x;q)_n(-1)^n q^{(n^2+3n)/2}}{(1-q)(q^2/z,zq,q^2/x,xq;q)_n}.$$

Therefore, dividing both sides by $(1-z)(1-q/z)$, and simplifying, we find that

$$\sum_{n=0}^{\infty}\frac{(x,q/x;q)_n q^n}{(z,q/z;q)_{n+1}}=\frac{(x,q/x;q)_\infty}{(q;q)_\infty^2}\sum_{n=0}^{\infty}\frac{(1-q^{2n+1})(-1)^n q^{(n^2+3n)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)}$$
$$=\frac{(x,q/x;q)_\infty}{(q;q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}q^{(n^2+7n+2)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)},\tag{5.3.7}$$

where in the last equality, we used the fact that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+3n)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)}$$
$$=\sum_{n=-\infty}^{-1}\frac{(-1)^{n+1}q^{(n^2+7n+2)/2}}{(1-q^{n+1}/z)(1-zq^n)(1-q^{n+1}/x)(1-xq^n)}$$

and that the series is absolutely convergent for $|q|<1$.

Finally, we invoke (5.3.6) for the series on the right side of (5.3.7). Upon simplification, we arrive at (5.1.5) and this completes the proof. □

Chapter 6

Open problems and further studies

6.1 Inequalities for the cumulation functions of k-ranks

First, we introduce the definition of the crank of a partition. Given a partition λ, let $o(\lambda)$ denote the number of ones in a partition, and define $\mu(\lambda)$ as the number of parts strictly larger than $o(\lambda)$. Then

$$\operatorname{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda, \text{ if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda), \text{ if } o(\lambda) > 0. \end{cases}$$

In [34], Garvan studied a partition statistic called the k-rank which generalized the rank and crank. For an integer t, he defined $N_k(t, N)$ by

$$\sum_{N\geq 0} N_k(t,N)q^N = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n((2k-1)n-1)/2+|t|n}(1-q^n). \tag{6.1.1}$$

When $k = 1$ (resp. $k = 2$), this is the generating function for the crank (resp. rank). For $k \geq 3$, the interpretation of $N_k(t, N)$ was also discussed in [34]. We define the *cumulation functions* of the k-rank by

$$\overline{N}_k(t,n) := \sum_{r\le -t} N_k(r,n) = \sum_{r\ge t} N_k(r,n) \quad \text{(by symmetry)}.$$

Computer evidence suggests the following conjecture.

Conjecture 6.1.1. *For all positive integers k, t, and n, we have*

$$\overline{N}_k(t,n) \ge \overline{N}_{k+1}(t,n). \tag{6.1.2}$$

An equivalent version of (6.1.2) with $k = 1$ was also conjectured by Andrews, Dyson and Rhoades [5] and proved by Chen, Q. Ji and W. J. T. Zang [26] combinatorially. In [60], we proved that (6.1.2) is true for fixed t and sufficiently large n. We are interested in providing a combinatorial proof of (6.1.2) for all positive integers n as in [26].

6.2 Nonnegativity of coefficients of certain Lambert series

As we mentioned in Chapter 2 and Chapter 3, computer evidence suggests the nonnegativity of the coefficients of the following generalized Lambert series,

$$\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{5n}},$$
$$\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}},$$
$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{10n}},$$
$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(q^{4n}-1)}{1+q^{10n}},$$

and

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{6n}}.$$

Is it possible to prove these statements without deriving dissections of these Lambert series? And are there more rank type generating functions with such a property?

6.3 Rank difference for overpartitions

An overpartition [29] is a partition in which the first occurrence of each distinct number may be overlined. There are two distinct ranks of interest: D-rank [51] and M_2-rank [52]. To define these two ranks, we use the notation $l(\cdot)$ to denote the largest part of an object, $n(\cdot)$ to denote the number of parts, and λ_0 for the subpartition of an overpartition consisting of the odd non-overlined parts. Then the D-rank of an overpartition λ is the largest part $l(\lambda)$ minus the number of parts $n(\lambda)$, a straight application of Dyson's partition rank to overpartitions. The M_2-rank of an overpartition λ is by

$$M_2\text{-rank}(\lambda) := \left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_0) - \chi(\lambda),$$

where $\chi(\lambda) = 1$ if $l(\lambda)$ is odd and non-overlined and $\chi(\lambda) = 0$ otherwise.

Rank differences between these ranks modulo 3 and 5 were proved by Lovejoy and Osburn [55,57]. Let $\overline{N1}(m,n)$ ($\overline{N2}(m,n)$, resp.) denote the number of overpartitions of n with D-rank (M_2-rank, resp.) m and $\overline{N1}(t,l,n)$ ($\overline{N2}(t,l,n)$, resp.) denote the number of overpartitions of n with D-rank (M_2-rank, resp.) congruent to t modulo l. Then their generating functions are

$$\overline{R1}(z,q) := \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{N1}(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-1;q)_n}{(zq,q/z;q)_n}$$

and

$$\overline{R2}(z,q) := \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{N2}(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{q^n(-1;q)_{2n}}{(zq^2,q^2/z;q^2)_n}.$$

Computer evidence suggests the following conjecture on the dissections of D-ranks modulo 3.

Conjecture 6.3.1. *For $\omega = e^{\pi i/3}$,*

$$\begin{aligned}&\sum_{n\geq 0}(\overline{N1}(0,6,n)+\overline{N1}(1,6,n)-\overline{N1}(2,6,n)-\overline{N1}(3,6,n))q^n\\ &=\overline{R1}(\omega,q)\\ &=\frac{J_{9,18}J_{18}^3}{J_{3,18}^2J_{6,18}}-2q\frac{J_{18}^3}{J_{3,18}J_{6,18}}+4q^2\frac{J_{18}^3}{J_{6,18}J_{9,18}}-2\frac{1}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n+2}}{1+q^{9n+3}}.\end{aligned}$$

It is an interesting question whether we could apply the method used in Chapter 2 and Chapter 3 to prove these dissections. Using the theory of weak Maass wave form, K. bringmann and Lovejoy obtained the following equalities between the two ranks of overpartitions modulo 2 (see [19, Corollary 1.2]):

$$\overline{N1}(0,2,4n+1)=\overline{N2}(0,2,4n+1)$$

and

$$\overline{N1}(0,2,4n+2)=\overline{N2}(1,2,4n+2),$$

for $n\geq 0$.

We hope to find proofs of the above relations applying the methods adopted by Lewis in [43, 44]. We are also interested in discovering relations between ranks of overpartitions for other modulo.

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