Issues in Robust and Networked Control Systems

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Statement of Originality

I hereby certify that the content of this thesis is the result of work done by me and has not been submitted for a higher degree to any other University or Institution.

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Summary

This thesis presents research results on control and filtering of discrete-time linear systems. The thesis is divided into two parts, the first is on control and filtering of uncertain linear systems with polytopic uncertainties and the second on control and filtering of networked control systems.

In the first part, we begin by considering a class of uncertain linear systems with polytopic uncertainties. Guaranteed cost control problem is addressed for such systems. Less conservative results than existing approaches are obtained based on the well-known linear matrix inequality (LMI) approach and the difference linear matrix inequality (DLMI) method. Then, we study the robust H_∞ and H_2 dynamic output feedback control of systems with polytopic uncertainties, which has not been studied in existing literature. We also investigate the H_2 and H_∞ filtering for discrete-time systems with polytopic uncertainties. By introducing more slack variables which offer additional flexibility in optimization than existing approaches, much less conservative results are obtained for designing filters. An iterative approach to further improve the filter performance is proposed and a detailed comparison has been made with existing results.

In the second part of this thesis, we study the designs of controller and filter for networked systems. We first investigate the control and filtering problem from the point of view of limited communication constraints in network. Based on a combined heuristic and convex optimization, we propose a procedure to compute a suboptimal communication sequence and the corresponding optimal controller or filter. The heuristic search method is much less computationally demanding than the existing exhaustive search method. We then consider the control problem of networked systems by focusing on network time delays. Here the plant is assumed to be discrete-time linear systems and the transmission delay is assumed to be random but bounded. We formulate the H_2 control problem for such system into a bilinear matrix inequality and then extend the sequential linear programming matrix method (SLPMM) for finding a sub-optimal fixed order controller. Finally, by employing the switched system theory, LMI based solutions are derived for networked control systems. Both the static and dynamic output feedback are considered.

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Symbols and Acronyms

$\mathcal{R}$	set of real numbers.
$\mathcal{Z}$	set of integer numbers.
$\mathcal{R}^n$	$n-$ dimensional real Euclidean space.
$\mathcal{R}^{n\times m}$	set of $n \times m$ real matrices.
I	identity matrix.
$I_{n\times n}$	identity matrix of dimension $n \times n$.
A^T	transpose of matrix A.
A^{-1}	inverse of matrix A.
A^{-T}	inverse $-$ transpose of matrix A.
x^T	transpose of vector x.
$A^{1/2}$	a symmetric square root of matrix $A = A^T \geq 0$, i.e. $A^{1/2} \cdot A^{1/2} = A$.
$diag\{A_1, A_2, \cdots, A_n\}$	block diagonal matrix with A_j being the diagonal entries.
$P{\geq}0$	symmetric positive semidefinite matrix $P \in \mathcal{R}^{n\times n}$.
$P{>}0$	symmetric positive definite matrix $P \in \mathcal{R}^{n\times n}$.
$P{\geq}Q$	$P{-}Q{\geq}0$ for symmetric matrices $P, Q \in \mathcal{R}^{n\times n}$.
$P{>}Q$	$P{-}Q{>}0$ for symmetric matrices $P, Q \in \mathcal{R}^{n\times n}$.
$\lambda(A)$	eigenvalues of matrix A.

$trace(A)$	trace operation of matrix $A \in \mathcal{R}^{n \times n}$.
$\mathcal{E}(J)$	expectation operation of cost J.
$det\{A\}$	determinant of A.
$\ell_2[0, \infty)$	space of square summable sequences on $[0, \infty)$ with value on $\mathcal{R}^n$.
$\|x\|_2$	ℓ_2 norm of $x \in [0, \infty)$.
$*$	symbol denotes the symmetric term within a matrix.
$A^{\perp}$	the bases of $\mathrm{Ker}(A)$.
ARE	Algebaric Riccati Inequality.
BMI	Bilinear Matrix Inequality.
DLMI	Difference Linear Matrix Inequality.
GCC	Guaranteed Cost Control.
IQC	Integral Quadratic Constraint.
LMI	Linear Matrix Inequality.
LQ	Linear Quadratic.
LTI	Linear Time Invariant.
NCS	Networked Control Systems.
RDE	Riccati Difference Equation.
SLPMM	Sequential Linear Programming Matrix Method.

Chapter 1

Introduction

1.1 Motivations

Linear estimation and control has important applications in engineering. For systems with known model and noise statistics, linear estimation with minimum error variance and linear quadratic Gaussian (LQG) control were the focus of research in the 1960s and the 1970s; see [AM79, KS72]. Realizing the difficulty of obtaining exact noise statistics, the so-called H_∞ filtering and control has attracted much interest of research in the past decades [FdSX92, DGKF89]. An H_∞ filter/controller does not require the knowledge of noise statistics and ensures the worst case performance. On the other hand, exact modeling of systems is usually difficult and system parameters may vary with time. This has motivated the study of robust estimation and control methods for systems with possible uncertainty [Zho98].

Robust control aims to ensure stability and system performance when a system is subjected to uncertainty of various forms. A great deal of efforts have been devoted to this study. In particular, the notion of quadratic stability, based on the search for a single Lyapunov function of the form $V = x^T Px$, has achieved considerable success

[Bar85], especially for systems with fast time-varying uncertainty. Among various quadratic stability based control and filtering techniques, the robust guaranteed cost control aims to design a controller that ensures both the robust stability and a guaranteed upper bound for a quadratic cost function and has attracted considerable interest; see [XS95, Gar97, Pet95, PM98] and the references therein. And filtering with guaranteed error variance for uncertain systems was first addressed in [Jai75] and further developed by many researchers, see [XdSF91, XSdS94, PM96, ZSX02, TS96].

In the above references, a Riccati equation based approach was adopted to deal with estimation and control of systems with parameter uncertainty of norm-bounded type. The results of these works involve searching for appropriate scaling parameters such that the associated algebraic Riccati equation (ARE) has a solution. This is not an easy task in general. Another drawback of the ARE based approach is that it assumes a fixed Lyapunov function for the entire family of systems characterized by the norm-bounded uncertainty.

An alternative based on the linear matrix inequality (LMI) approach has gained popularity since the development of the interior point algorithm for convex optimization. In [LF97, dST99, Ger99, GdO98, FdSL99, Nic98, dSL99, dST99], the LMI approach has been applied to solve the robust H_2 or H_∞ filtering/control for systems with uncertainty of norm-bounded or polytopic type or uncertainty satisfying integral quadratic constraint (IQC). These works do not require searching for scaling parameters but still apply a fixed Lyapunov function.

While the notion of quadratic stability has been effective and convenient in dealing with parametric uncertainty, it may be overly conservative, especially for slowly time-varying or constant uncertainty as it employs a fixed Lyapunov function for all uncertainties. To reduce the design conservatism, there have been many attempts in the past few years. In [BdST02, MK00, FAG96], a Lyapunov function which is a linear or quadratic function of the uncertain parameter vector has been developed. In [dOG99, GBGdO00], an additional parameter-dependent matrix variable has been introduced which allows the Lyapunov function to be vertex-dependent. This technique has been effective in alleviating the design conservatism significantly. Indeed, a less conservative design using this technique has been recently demonstrated in [SXS01, GdOB02, dOGB99]. A further improved robust stability condition was reported in [PABB00] which contains the condition of [dOG99] as a special case.

Motivated by these observations, in part I of this thesis, we will derive some new results to further reduce the conservatism in robust guaranteed cost control and robust H_2/H_∞ filtering for discrete-time linear systems with ploytopic uncertainties.

On the other hand, the development in communication technology has made it popular to use networks in monitoring and control systems, especially in those that are large scale and physically distributed, or those that require extensive cabling. It can greatly decrease the hardwiring and reduce the cost of installation and implementation. The more modular and flexible structure of networked systems makes it much easier for removal, exchange, and adding parts.

The networks for control are different from data networks. In data networks, large data packets are sent out occasionally at high data rates in a short time; while in control networks data are continuously transmitted at relatively constant data rates [LMT01b]. In addition, control networks have a fixed sampling period in general and can meet the crucial time-critical requirement. The primary objective of control networks is to efficiently exploit the available communication resource to optimize system performance.

There are mainly two issues inherent to network-based systems. One is the network-induced delays when exchanging data among devices. These delays are either constant, time-varying or even random. Data packets can not only suffer from transmission delay but also be lost during transmission. The other is the limitation on bandwidth as in any kind of networks. This imposes a constraint on the availability and accuracy of data that are used for decision making. Clearly, the presence of communication network complicates the application of standard results of control and filtering theory. Conventional control theories such as synchronized control and non-delayed sensing and actuation must be re-evaluated prior to their application to network-based systems.

The above presents new challenges to control engineers and brings forth a lot of efforts to investigate the effects of finite communication rates and time delay in control problems. Analysis and design of network-based systems have just started to develop and are new to the control community, although they have been used in industry and are now widely supported from a device level to a system level. It is of theoretical as well as practical interest to study this area.

Modeling and analysis of control networks are related to that of switched systems. In [BBW98], an equivalence between a control network and a set of switched systems is established, which offers a way to model, analyze and design control networks. A stabilizing controller for a network-based system is designed by studying the stability of a corresponding switched system. In [IF02], a similar method is adopted to design a switching controller which can stabilize a linear continuous-time plant. In particular, considering the limited data rates in control networks a dwell-time scheme is employed to prevent the actuator from fast switching which can not only increase the necessary data rates but also may damage the system. Several models for systems with time-variant delays are investigated in [Sic99].

Many papers have devoted to the study of stability and performance of network-based systems. In [XJH+03], the problem of designing a networked controller and allocating the communication resources to optimize the performance is addressed. Since the problem is in general not convex, an iterative heuristic method is proposed for the joint design problem. In [ZM03], a systematic controller design method is presented, by which the problem of the stabilization of NCS is reduced to that of an asymptotically stable observer design for linear systems with missing measurements. In [IF00] the well known lifting technique is used to clarify the advantage of using networks in the context of decentralized control. It is shown that information exchange among local controllers through a network can enlarge the class of plants to be stabilized.

The problem of the stabilization of linear systems with limited information has also attracted much interest recently. Both quantized state feedback controller and estimator have been derived for SISO discrete-time systems in [EM01]. The quantizer is so-called logarithmic for a quadratically stabilizable system and the coarsest quantization is given in terms of the system's unstable poles. The method is generalized to the case of two-input systems [EF02] and multi-input systems [KV02]. Further, the problem is reinterpreted in [FX03] by using the sector bound method, which allows the result to be generalized to various cases: output feedback control, MIMO systems and control with performance.

Meanwhile many researchers have dealt with time-delay in control networks. Nilsson [Nil98] analyzes network-based systems in discrete-time domain. He further tries to model the network delay as constant delays, independent random delays, and random delays governed by underlying Markov chains. From there, he solves the LQG optimal control problem for the different delay models. In [WHB99], the notion of maximum allowable transfer interval (MATI) is introduced and the controller is designed first without considering the presence of the network in the feedback loop. Then the goal is to find the maximum value of MATI for which the desired performance of a linear system is guaranteed to be retained. Subsequently, this method is extended to deal with nonlinear networked systems in [WBB99a]. In [ZB01], network-based model with network-induced delay is presented; also the stability problem is analyzed by using stability regions; and the case of data packet dropout is considered. Subsequently, the problem of data dropout has been further investigated in [LL02, LL03, HT02, SS01]. Especially, the effect of communication packet losses in the vehicle control problems where information is exchanged via a local area network is studied in [SS01]. By applying the results for discrete-time linear systems with Markovian jumping parameters, a controller is constructed such that the closed-loop system is mean square stable for a given packet loss rate.

To improve the performance for network-based systems, many methods have been proposed. In [WHB99, WBB99b], a novel scheduling policy called try-once-discard (TOD) is introduced and compared with static scheduling. It shows that the dynamic TOD scheduler is superior to static scheduler in terms of improving closed-loop performance. In [BW00], the authors try to estimate the state of a plant in between two transmission time instants instead of using piecewise inputs for the controller. Simulations show that the performance of the closed-loop system is improved in the presence of predictors. An asynchronous cooperative resource allocation strategy is introduced in [GP03], which involves deciding how to divide a limited resource across multiple demands. It is shown that the proposed strategy can optimize the design objectives.

At the same time, many schemes are developed to reduce network traffic. Estimators are used in [YTS02] at each node to estimate the values of the other outputs at other nodes. The estimated values are then used to compute the control law at each node. Only when the estimated value deviates from the true value by more than a pre-defined tolerance, is the actual value broadcast to the rest of the system. In [OMT02], a “deadband” is used on a node, with which the node does not broadcast a new message if the node signal is within the deadband. Thus a significant saving in bandwidth is achieved to allow more resources to utilize the network. And the reduction of data traffic in network is fulfilled by making full use of the knowledge of the plant dynamics in [MA02]. A mathematical framework for analyzing the effect of time-delays on the performance of distributed control systems is presented in [YTS01]. It is shown that different implementation architectures of distributed systems can lead to different time-delays. A function of performance degradation is defined first and then it can be easily compared for all the cases of implementation.

Motivated by the aforementioned works, we study the networked control systems in Part II from the point of view of communication constraints and time-delay.

1.2 Objectives

The main objectives of the thesis are as follows:

- To develop less conservative guaranteed cost controllers for discrete-time uncertain linear systems based on the LMI approach.
- To solve the problem of dynamic output feedback controller design for linear discrete-time systems with polytopic uncertainties.
- To present less conservative designs of H_2 and H_∞ filters for discrete-time linear systems with polytopic uncertainty.
- To develop methodologies for H_2 and H_∞ filter/controller design for networked systems with limited communication constraints.
- To study the problems of stabilization and H_2 controller design for networked control systems with random but bounded time delays.

1.3 Major Contributions of the Thesis

Major contributions of the thesis are summarized as follows:

- Less conservative GCC controllers are studied for discrete-time uncertain systems based on the LMI and DLMI approaches. Both the state-feedback and output-feedback cases are investigated.

- Robust H_∞ and H_2 dynamic output feedback controllers are designed for discrete-time systems with polytopic uncertainties.

- Improved robust H_2 and H_∞ filtering techniques are developed for discrete-time systems with polytopic uncertainties. An iterative algorithm is proposed to further refine the suboptimal filter.

- Robust H_∞ and H_2 filtering are addressed for discrete-time systems with limited communication constraints. A heuristic search method is given to obtain a suboptimal communication sequence and the corresponding optimal filter parameters.

- The optimal control of networked systems is solved via a combined heuristic and convex optimization approach, by which an explicit expression for the controller is provided in terms of the solution of the LMIs. A suboptimal communication sequence and the corresponding optimal controller are obtained simultaneously.

- An iterative LMI algorithm is presented for designing H_2 controllers for discrete-time networked control systems with random but bounded delays in the feedback loop. The SLPMM ([Lei01]) method is extended to obtain the controller parameters.

- Less conservative stabilization conditions are given for networked control systems with random but bounded delays via a convex optimization approach. The cases of static feedback and dynamic feedback are both considered.

1.4 Organization of The Thesis

The rest of the thesis is organized as follows:

Part I includes chapters 2-4 and it focuses on robust control and filtering for discrete-time systems with polytopic uncertainties. First guaranteed cost controllers for uncertain systems are obtained. Then in Chapter 3 the results on the design of dynamic output feedback controller for systems with polytopic uncertainties are proposed. Less conservative designs of H_2 and H_∞ filters are given in Chapter 4. The details of each chapter are as follows:

> **Chapter** 2 studies the problem of robust guaranteed cost control for discrete-time systems with polytopic uncertainties. A quadratic cost function is considered as a performance measure for the closed-loop system. In the infinite horizon case, sufficient conditions are provided to construct a controller that places the closed-loop poles in a specified disk and guarantees a cost upper bound for all admissible uncertainties. These conditions are expressed in terms of linear matrix inequalities. In the finite horizon case, we propose a guaranteed cost control design for uncertain linear time-varying systems based on the DLMI approach.
>
> **Chapter** 3 is devoted to the design of H_2 and H_∞ controllers for discrete-time systems. The case of dynamic output feedback is studied and the system under consideration is subject to polytopic uncertainties. A unified treatment of designing H_2 and H_∞ controller is proposed. The controller parameters are given in terms of a linear matrix inequality which can be solved efficiently by convex optimization.
>
> **Chapter** 4 is concerned with the robust H_2 and H_∞ filtering problems for linear discrete-time systems with polytopic parameter uncertainty. It is shown that a more efficient evaluation of robust H_2 or H_∞ performance can be obtained by a matrix inequality condition which contains additional free parameters as compared to existing characterizations. When applying this new matrix inequality condition to the robust filter design, these parameters provide extra freedoms in optimizing the guaranteed H_2 or H_∞ performance and

lead to a less conservative design. We also propose an iterative algorithm to further refine the suboptimal filter.

Part II includes chapters 5-8 and it focuses on control and filtering for networked systems. First we consider the robust filtering problem for uncertain discrete-time systems with limited communication constraints in Chapter 5. And the methodology is extended to the controller design in Chapter 6. An H_2 controller design method is presented in Chapter 7 for networked control systems with random but bounded delays. Less conservative stabilization conditions are given in Chapter 8. The details of each chapter are as follows:

Chapter 5 jointly considers robust filtering and communication issues for systems with distributed sensors. The limited bandwidth in networks is taken into account when investigating filtering problems under a scheduled releasing policy. Given an uncertain system with polytopic uncertainty and limited communication resource, our objective is to find an optimal communication sequence and a robust optimal filter under the H_∞ or H_2 performance. A heuristic search method is proposed to seek a sub-optimal communication sequence, which greatly reduces the computational cost as compared with the existing exhaustive search method.

Chapter 6 discusses the optimal H_2 and H_∞ control problems for networked systems with limited communication constraint. For a given communication sequence, the problem is formulated into a periodic control problem for which a direct LMI design method is developed. We also propose a heuristic search method for seeking a sub-optimal communication sequence, which in conjunction with the convex optimization gives a solution to the optimal limited communication control problem. As compared with the exhaustive search of communication sequence, our approach greatly reduces the computational cost.

Chapter 7 deals with the problem of designing an H_2 controller for a networked control system (NCS) with communication delays from the sensor to the controller and/or from the controller to the plant. Our objective is to design a robust controller that will not only stabilize the system but also achieve a sub-optimal H_2 performance in the face of possible communication delays. The feedback control problem for the original system is first converted to a static output feedback control problem. A recursive LMI algorithm is then presented to compute an output feedback H_2 controller for the system. Our approach allows a fixed order controller.

Chapter 8 investigates the problem of stabilization of NCS. Switched system theory is employed and LMI based solutions are derived for the cases of static and dynamic output feedback. The proposed method avoids solving the equality constraints or using the iterative approach which are common in dealing with the stabilization of NCS. The controller parameters can be obtained in terms of linear matrix inequalities.

Chapter 9 concludes the thesis and provides some possible directions for further research.

PART I

UNCERTAIN LINEAR SYSTEMS

Chapter 2

Robust Guaranteed Cost Control of Discrete-time Systems with Polytopic Uncertainties

2.1 Introduction

Robust guaranteed cost control (GCC) aims to design a controller that guarantees an upper bound for a quadratic cost function while ensuring the robust stability of the closed-loop system. Most of the existing studies investigated the norm-bounded type of uncertainties [XS95,Pet95] by using fixed Lyapunov functions. Very recently, robust control has been addressed for systems with polytopic uncertainty which is more flexible in characterizing uncertainty of practical systems [LSY99, AT00, dOGH99, SXS01]. Generally speaking, the latter uncertainty characterization leads to a less conservative control design than the former as it exploits some structure information of the uncertainty [LSY99, SXS01, GdOB02]. Besides, it allows us to introduce different Lyapunov functions for each different point of the uncertainty polytope.

While the guaranteed cost control ensures the cost of the system to be within a certain bound for all admissible uncertainties, it provides little control over the transient response. One way to take transient performance into account is to specify the locations of the poles of the closed-loop system. When there are no uncertainties in the system, it is possible to exactly place closed-loop poles at the specified locations. In the presence of uncertainty, however, one can only place the poles within some regions. Thus in order to simultaneously tune the cost performance and the transient behavior of the uncertain system, the robust guaranteed cost control and pole-placement constraint are combined together; see [MP96] for the continuous-time case and [Gar97] for the discrete-time one.

In this chapter we extend the method in [LSY99] to study the problem of robust guaranteed cost control with pole-placement constraint. Unlike [Gar97] and [MP96] where the norm-bounded uncertainty is tackled, we consider a class of convex-bounded uncertain discrete-time systems. Based on parameter-dependent Lyapunov functions we tackle this problem in terms of LMIs and obtain less conservative results than the existing works. In the finite horizon case, we employ the recently developed DLMI method to solve the guaranteed cost control problem. The DLMI approach gives the least conservative control design as demonstrated in the given numerical examples.

2.2 The Infinite Horizon Case

2.2.1 Problem Formulation and Preliminaries

Consider the linear discrete-time system described by

$$x(k+1) = Ax(k) \tag{2.2.1}$$

where $x(k) \in \mathcal{R}^n$ is the state vector and $A \in \mathcal{R}^{n \times n}$ is the state matrix which is uncertain.

Associated with this system is the cost function:

$$J = \sum_{k=0}^{\infty} x^T(k) Q x(k), \; Q^T = Q > 0. \tag{2.2.2}$$

Lemma 2.2.1 *[GB95] Given matrix $A \in \mathcal{R}^{n \times n}$, the eigenvalues of matrix A are located within the specified disk region with center $\eta + j0$ and the radius ρ, i.e. $\lambda(A) \subset \mathcal{D}(\eta, \rho)$, if and only if there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} -\rho^2 P & (A - \eta I)^T \\ (A - \eta I) & -P^{-1} \end{bmatrix} < 0. \tag{2.2.3}$$

Definition 2.2.1 The system (2.2.1) is said to be $\mathcal{D}$ stable with guaranteed cost if there exists a positive definite matrix $P \in \mathcal{R}^{n \times n}$ such that

$$(A - \eta I)^T P (A - \eta I) - \rho^2 (P - Q) < 0. \tag{2.2.4}$$

In this situation, the matrix P is called the $\mathcal{D}$ cost matrix.

Using this definition one can state the following theorem which shows that $\mathcal{D}$ stablility with guaranteed cost of a system implies that the system meets the guaranteed cost bound and the pole-placement constraint simultaneously.

Theorem 2.2.1 *Given real scalars $\eta \in (-1, 1)$ and $0 < \rho < 1 - |\eta|$, if the system (2.2.1) is $\mathcal{D}$ stable with guaranteed cost, then all the poles of the system lie in the circular region $\mathcal{D}(\eta, \rho)$ and the value of the cost function satisfies*

$$J < \frac{1 - |\eta|}{\rho^2} x^T(0) P x(0), \; x(0) \neq 0. \tag{2.2.5}$$

Proof From Definition 2.2.1, it is known that if the system (2.2.1) is $\mathcal{D}$ stable with guaranteed cost then there exists a matrix P such that (2.2.4) holds, thus we have

$$(A-\eta I)^T P(A-\eta I)-\rho^2 P<-\rho^2 Q<0.$$

By using Lemma 2.2.1, it is easy to know $\lambda(A)\subset\mathcal{D}(\eta,\rho)$. To show the guaranteed cost bound, we first consider the case $\eta<0$. In this case, using the fact that

$$(A+I)^T P(A+I)\geq 0$$

we have for all $\eta<0$

$$-\eta PA-\eta A^T P\geq\eta A^T PA+\eta P.$$

Again from the above inequality and (2.2.4), we have

$$A^T PA+\eta^2 P+\eta A^T PA+\eta P-\rho^2 P+\rho^2 Q<0.$$

That is

$$(1+\eta)A^T PA-(1+\eta)P+[(1+\eta)^2-\rho^2]P+\rho^2 Q<0.$$

Since $1+\eta\geq\rho$, we have

$$A^T PA-P+\rho^2(1+\eta)^{-1}Q<0. \tag{2.2.6}$$

Now we select the Lyapunov function $V(x(k))=x^T(k)Px(k)$. Then we have

$$V(x(k+1))-V(x(k))=x^T(k)(A^T PA-P)x(k)<-x^T(k)\rho^2(1+\eta)^{-1}Qx(k)$$

for nonzero $x(k)$. Since there exists some k such that $x(k)\neq 0$, we have

$$\frac{\rho^2}{1+\eta}\sum_{k=0}^{\infty}x^T(k)Qx(k)<V(x(0))-V(x(\infty)).$$

Because the system (2.2.1) is quadratically stable, $x(\infty) \to 0$. Hence it is clear that

$$J < \frac{1+\eta}{\rho^2} V(x(0)) = \frac{1+\eta}{\rho^2} x^T(0) P x(0).$$

For the case of $\eta > 0$, the proof can be completed in much the same way as that for $\eta < 0$.

Remark 2.2.1 From (2.2.5) we can see that the cost depends not only on the initial state value and the $\mathcal{D}$ cost matrix P but also on η and ρ. When $\eta = 0$ and $\rho = 1$, the inequality (2.2.4) becomes $A^T PA - P + Q < 0$ which is related to the standard guaranteed cost control problem.

Next we consider the uncertain discrete-time linear system:

$$x(k+1) = (A + \Delta A)x(k) + (B + \Delta B)u(k) \quad (2.2.7)$$

$$y(k) = (C + \Delta C)x(k) \quad (2.2.8)$$

where $x(k) \in \mathcal{R}^{n\times n}$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^r$ is the measurement. A, B, C are real constant matrices with appropriate dimensions. ΔA, ΔB, ΔC are unknown matrices which are known to lie in the following uncertainty polytope:

$$\Omega \triangleq \left\{ (\Delta A, \Delta B, \Delta C) | (\Delta A, \Delta B, \Delta C) = \sum_{i=1}^{M} \theta_i (\bar{A}^{(i)}, \bar{B}^{(i)}, \bar{C}^{(i)}), \theta_i \geq 0, \sum_{i=1}^{M} \theta_i = 1 \right\}$$

where $M = 2^l$ and l is the total number of the uncertain scalar parameters in ΔA, ΔB and ΔC.

From the above definition, the system can be rewritten as

$$x(k+1) = A(\theta)x(k) + B(\theta)u(k) \quad (2.2.9)$$

$$y(k) = C(\theta)x(k) \quad (2.2.10)$$

where $A(\theta) = \sum_{i=1}^{M} \theta_i A^{(i)}$ and $A^{(i)} = A + \bar{A}^{(i)}$; $B(\theta) = \sum_{i=1}^{M} \theta_i B^{(i)}$ and $B^{(i)} = B + \bar{B}^{(i)}$; $C(\theta) = \sum_{i=1}^{M} \theta_i C^{(i)}$ and $C^{(i)} = C + \bar{C}^{(i)}$.

Associated with the uncertain system (2.2.7) is the following cost function:

$$J = \sum_{k=0}^{\infty} (x^T(k)Qx(k) + u^T(k)Ru(k)), \;\; Q^T = Q > 0, \;\; R^T = R > 0. \tag{2.2.11}$$

Definition 2.2.2 The uncertain system (2.2.7) is called $\mathcal{D}$ stabilizable with guaranteed cost if there exists a control u such that the corresponding closed-loop system is $\mathcal{D}$ stable with guaranteed cost. And the control law is called the robust guaranteed cost control (GCC).

Thus the problem to be dealt with in this chapter is to find a guaranteed cost control u to make the system (2.2.7) satisfy the following performance simultaneously for any uncertainty from the polytope:

(1) all the closed-loop poles of the system (2.2.7) lie in a given circular region $\mathcal{D}(\eta, \rho)$.

(2) the cost function (2.2.11) satisfies some bound $J \leq J_0$.

2.2.2 Main Results

In this subsection we assume that all the state variables are measurable and want to design a guaranteed cost control $u(k) = Kx(k)$ such that the constraints (1) and (2) are satisfied at the same time. By using the state feedback control law, the closed-loop system can be described by the following equations:

$$x(k+1) = A_c(\theta)x(k) \tag{2.2.12}$$

$$y(k) = C(\theta)x(k) \tag{2.2.13}$$

where $A_c(\theta) = \sum_{i=1}^{M} \theta_i (A^{(i)} + B^{(i)} K)$. And the cost function (2.2.11) is changed into

$$J = \sum_{k=0}^{\infty} x^T(k)(Q + K^T RK)x(k). \tag{2.2.14}$$

Theorem 2.2.2 *Given real scalars η and ρ satisfying $\eta \in (-1, 1)$ and $0 < \rho < 1 - |\eta|$, if there exist matrices $P^{(i)} = (X^{(i)})^{-1}$, $i = 1, 2, \cdots, M$, Y, V and $Z \in \mathcal{R}^{n \times n}$ such that*

$$\begin{bmatrix} -\rho^2(Z + Z^T - X^{(i)}) & * & * & * \\ (A^{(i)} - \eta I)Z + B^{(i)} Y & -X^{(i)} & * & * \\ Q^{1/2} Z & 0 & -\rho^{-2} I & * \\ R^{1/2} Y & 0 & 0 & -\rho^{-2} I \end{bmatrix} < 0 \tag{2.2.15}$$

$$\begin{bmatrix} -V & I \\ I & -X^{(i)} \end{bmatrix} < 0 \tag{2.2.16}$$

then the system is $\mathcal{D}$ stabilizable with guaranteed cost and the control law given by $K = YZ^{-1}$ will achieve

$$\lambda(A_c) \subset \mathcal{D}(\eta, \rho)$$

$$J < \frac{1 - |\eta|}{\rho^2} x^T(0) V x(0). \tag{2.2.17}$$

Proof From Definitions 2.2.1 and 2.2.2 we know that the uncertain system (2.2.7) is $\mathcal{D}$ stabilizable with guaranteed cost if the following inequality

$$A_{c\eta}^T(\theta) P(\theta) A_{c\eta}(\theta) - \rho^2 P(\theta) + \rho^2 (Q + K^T RK) < 0 \tag{2.2.18}$$

is satisfied, where $A_{c\eta}(\theta) = A_c(\theta) - \eta I$ with $A_c(\theta)$ given in (2.2.12) and $P(\theta)$ depends on the parameter vector θ. Thus $V(x(k)) = x^T(k) P(\theta) x(k)$ is a parameter-dependent Lyapunov function.

Using the Schur complement, we obtain from (2.2.18) that

$$\begin{bmatrix} -\rho^2 P(\theta) & A_{c\eta}^T(\theta) & Q^{1/2} & K^T R^{1/2} \\ A_{c\eta}(\theta) & -P^{-1}(\theta) & 0 & 0 \\ Q^{1/2} & 0 & -\rho^{-2} I & 0 \\ R^{1/2} K & 0 & 0 & -\rho^{-2} I \end{bmatrix} < 0.$$

Pre-multiplied by $diag\{P^{-1}(\theta), I, I, I\}$ and post-multiplied by $diag\{P^{-1}(\theta), I, I, I\}$, the above inequality is equivalent to

$$\begin{bmatrix} -\rho^2 X(\theta) & X(\theta) A_{c\eta}^T(\theta) & X(\theta) Q^{1/2} & X(\theta) K^T R^{1/2} \\ A_{c\eta}(\theta) X(\theta) & -X(\theta) & 0 & 0 \\ Q^{1/2} X(\theta) & 0 & -\rho^{-2} I & 0 \\ R^{1/2} K X(\theta) & 0 & 0 & -\rho^{-2} I \end{bmatrix} < 0 \qquad (2.2.19)$$

where $P^{-1}(\theta) = X(\theta)$. Now, consider the set of LMIs:

$$\begin{bmatrix} -\rho^2 X^{(i)} & X^{(i)}(A^{(i)} + B^{(i)} K - \eta I)^T & X^{(i)} Q^{1/2} & X^{(i)} K^T R^{1/2} \\ (A_i + B^{(i)} K - \eta I) X^{(i)} & -X^{(i)} & 0 & 0 \\ Q^{1/2} X^{(i)} & 0 & -\rho^{-2} I & 0 \\ R^{1/2} K X^{(i)} & 0 & 0 & -\rho^{-2} I \end{bmatrix} < 0 \qquad (2.2.20)$$

for $i = 1, 2, \cdots, M$, where $A^{(i)}$ and $B^{(i)}$ are the matrices given in (2.2.9).

The inequalities of (2.2.20) are not convex. The common method to solve this non-convexity is to replace $X^{(i)}$, $i = 1, 2, \cdots, M$, by the same matrix X. Obviously this is the quadratic stability requirement, which is well known for leading to significant conservativeness. To reduce the conservatism we will use the method of [LSY99], which relaxes the requirement on the quadratic stability while preserving the convexity inside the uncertain polytope. In view of Lemma 2.2 in [LSY99], it can be shown that (2.2.20) holds for some $X^{(i)} > 0$ if and only if the following inequalities

$$\begin{bmatrix} -\rho^2(Z+Z^T-X^{(i)}) & Z^T(A^{(i)}+B^{(i)}K-\eta I)^T & Z^TQ^{1/2} & Z^TK^TR^{1/2} \\ (A_i+B^{(i)}K-\eta I)Z & -X^{(i)} & 0 & 0 \\ Q^{1/2}Z & 0 & -\rho^{-2}I & 0 \\ R^{1/2}KZ & 0 & 0 & -\rho^{-2}I \end{bmatrix} < 0, \tag{2.2.21}$$

$i = 1, 2, \cdots, M$, are satisfied for some matrices $Z \in \mathcal{R}^{n\times n}$ and $X^{(i)} > 0$. Now the inequalities (2.2.21) are convex in $X^{(i)}$, which implies that if (2.2.21) holds for all the vertices, it holds for any point within the polytope.

In (2.2.21), letting $Y = KZ$ we can obtain (2.2.15). Note from (2.2.15) that $Z+Z^T > X^{(i)} > 0$, which implies that Z is invertible.

From the proof above we can see clearly that $\lambda(A_c) \subset \mathcal{D}(\eta,\rho)$, since (2.2.15) implies (2.2.18). As for the guaranteed cost bound, it is easy to verify that for each i, $i = 1, 2 \cdots M$, the corresponding upper bound is $\frac{1-|\eta|}{\rho^2}x^T(0)P^{(i)}x(0)$. Thus for all i, $i \in \{1, 2 \cdots M\}$, the upper bound is $\frac{1-|\eta|}{\rho^2}\max_i x^T(0)P^{(i)}x(0)$. From (2.2.16) we know $V > P^{(i)} = (X^{(i)})^{-1}$, $\forall i \in \{1, 2 \cdots M\}$ and then we have $J < \frac{1-|\eta|}{\rho^2}x^T(0)Vx(0)$. This completes the proof.

Remark 2.2.2 It can be seen that when $X^{(i)} = X$ and $Z = Z^T = X$, (2.2.21) reduces to (2.2.20) with $X^{(i)} = X$. This indicates that the maximum of $trace((X^{(i)})^{-1})$ is less than or equal to that of $trace(X^{-1})$. So the purpose for reducing the conservatism can be achieved.

When $\eta = 0$, $\rho = 1$ the problem of $\mathcal{D}$ stabilizability with guaranteed cost reduces to the guaranteed cost control problem. For this we give the following theorem.

Theorem 2.2.3 *If there exist matrices* $P^{(i)} = (X^{(i)})^{-1} > 0$, $i = 1, 2 \cdots M$, Y, Z

and $V \in \mathcal{R}^{n\times n}$ such that

$$\begin{bmatrix} X^{(i)} - Z - Z^T & Z^T A^{(i)T} + Y^T B^{(i)T} & Z^T Q^{1/2} & Y^T R^{1/2} \\ A^{(i)} Z + B^{(i)} Y & -X^{(i)} & 0 & 0 \\ Q^{1/2} Z & 0 & -I & 0 \\ R^{1/2} Y & 0 & 0 & -I \end{bmatrix} < 0 \qquad (2.2.22)$$

$$\begin{bmatrix} -V & I \\ I & -X^{(i)} \end{bmatrix} < 0, \qquad (2.2.23)$$

then under the control law $K = YZ^{-1}$ the cost bound satisfies

$$J < x^T(0)Vx(0). \qquad (2.2.24)$$

Proof The proof is much the same as that of Theorem 2.2.2, thus omitted.

Therefore the problem of $\mathcal{D}$ stabilizability with guaranteed cost can be considered as an LMI feasibility problem. This problem is convex and can be efficiently solved using the corresponding LMI software [GNLC95]. In the following we will study the optimization problem in (2.2.17) or in (2.2.24). Clearly the bound in (2.2.17) or in (2.2.24) depends on the initial condition $x(0)$. This dependence can be removed by assuming that $x(0)$ is a zero mean random variable satisfying $\mathcal{E}(x(0)x^T(0)) = I$, where $\mathcal{E}$ denotes the mathematical expectation. Thus the cost function bound given by (2.2.17) and (2.2.24) can be rewritten as $\mathcal{E}(J) < \frac{1-|\eta|}{\rho^2} trace(V)$ and $\mathcal{E}(J) < trace(V)$ respectively.

Remark 2.2.3 Theorem 2.2.2 or 2.2.3 suggests a corresponding upper bound for the cost function. Note that the above cost bound is a linear and convex function of the matrix $P^{(i)}$. Indeed, the problem of finding the minimum upper bound for the cost function J will be an LMI eigenvalue problem. This optimization problem is stated as follows:

$$minimize \quad trace(V) \qquad (2.2.25)$$

$$subject\ to \quad (2.2.15), (2.2.16)\ or\ (2.2.22), (2.2.23). \tag{2.2.26}$$

Although Theorem 2.2.2 and Theorem 2.2.3 are based on a relaxed quadratic stability requirement, they are still somehow conservative because Z is still constant. Assuming (2.2.15) or (2.2.22) has a solution that minimizes $trace(V)$, then the upper guaranteed cost bound of the uncertain system can be further reduced by the following method similar to that of [SXS01]. First we give a lemma which will play a key role in the sequel to reduce conservatism.

Lemma 2.2.2 *[SXS01] Given symmetric matrices P and X such that*

$$P + X > 0,\ and\ det\{P\} \neq 0.$$

Then the following inequality is satisfied

$$(P + X)^{-1} \geq P^{-1} - P^{-1} X P^{-1}.$$

From Theorem 2.2.2 and its proof, it can be seen that to obtain the guaranteed cost bound we can minimize $trace(V)$ subject to (2.2.16) and

$$\begin{bmatrix} -\rho^2 (X^{(i)})^{-1} & (A^{(i)} + B^{(i)} K - \eta I)^T & Q^{1/2} & K^T R^{1/2} \\ A^{(i)} + B^{(i)} K - \eta I & -X^{(i)} & 0 & 0 \\ Q^{1/2} & 0 & -\rho^{-2} I & 0 \\ R^{1/2} K & 0 & 0 & -\rho^{-2} I \end{bmatrix} < 0 \tag{2.2.27}$$

which is obtained by pre- and post-multiplying (2.2.20) by $diag\{(X^{(i)})^{-1}, I, I, I\}$. However, the matrix inequality (2.2.27) is bilinear in $X^{(i)}$ and feedback K. This can be solved by using Lemma 2.2.2. Applying Lemma 2.2.2 to (2.2.20) and (2.2.16), we can see that a solution to (2.2.20) and (2.2.16) exists if there exist $X^{(i)}$ and $F^{(i)}$

such that

$$\begin{bmatrix} -\rho^2((X^{(i)})^{-1}-(X^{(i)})^{-1}F^{(i)}(X^{(i)})^{-1}) & * & * & * \\ A^{(i)}+B^{(i)}K-\eta I & -X^{(i)}-F^{(i)} & * & * \\ Q^{1/2} & 0 & -\rho^{-2}I & * \\ R^{1/2}K & 0 & 0 & -\rho^{-2}I \end{bmatrix} < 0 \quad (2.2.28)$$

$$\begin{bmatrix} -V & I \\ I & -F^{(i)}-X^{(i)} \end{bmatrix} < 0 \quad (2.2.29)$$

Inequality (2.2.28) is still not linear in $X^{(i)}$, but it has the advantage of being solved by iterations. As in [SXS01], the following iterative procedure is given to find $X^{(i)}$ and feedback gain K satisfying (2.2.28) and (2.2.29).

The Iterative Algorithm:

- Step 1: Obtain the initial values $X_0^{(i)}$ by a linear combination of $X^{(i)}$ derived for all the vertex points by using Theorem 2.2.2.
- Step 2: At the k-th iteration step, based on the $X_{k-1}^{(i)}$ found previously, solve (2.2.28) and (2.2.29) to obtain $F_k^{(i)}$, feedback gain K_k and a minimum value of $trace(V_k)$.
- Step 3: Calculate $X_k^{(i)} = X_{k-1}^{(i)} + F_k^{(i)}$.
- Step 4: If $|trace(V_k) - trace(V_{k-1})| < \epsilon$ fails, where ϵ is a predefined convergence tolerance, then go to step 2 and increase k by one.
- Step 5: When the tolerance is satisfied, the guaranteed cost bound can be achieved by $trace(V_k)$, and $u = K_k x$ is a guaranteed cost control.

In the iterative algorithm above, the trace of V_k is minimized at each step. So it can be inferred that $trace(V_k)$ should be non-increasing and bounded from below. Hence it should converge and provide a guaranteed cost bound which is less than or equal to the value obtained by Theorem 2.2.2. This will be shown in the sequel via an example. Thus the iterative algorithm and Theorem 2.2.2 jointly introduce a robust guaranteed cost controller design method which will be much less conservative than the previous approaches.

2.3 The Finite Horizon Case

In [XS95], the finite horizon case is considered based on the recursive Riccati difference equation (RDE) for the norm-bounded uncertainty. In this section, we will apply a recently developed method called difference LMI (DLMI) to tackle the GCC problem without the pole-placement constraint. For convenience, we first give some results about GCC in the case of finite horizon. Consider the discrete linear time-varying system:

$$x(k+1) = A_k x(k) + B_k u(k), \; k = 0, 1, \cdots, N-1 \tag{2.3.1}$$

$$y(k) = C_k x(k) \tag{2.3.2}$$

where $x(k) \in \mathcal{R}^{n\times n}$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^r$ is the measurement. A_k, B_k, C_k are real matrices with appropriate dimensions. For this system we define the following cost function:

$$J = x^T(N)Sx(N) + \sum_{k=0}^{N-1}(x^T(k)Qx(k) + u^T(k)Ru(k)). \tag{2.3.3}$$

We recall Definition 2.1 and Theorem 2.1 in [XS95] below.

Definition 2.3.1 [XS95] A control law $u(k) = K_k x(k)$ is said to be a GCC associated with a cost matrix P_0 for the system (2.3.1) if there exists a positive definite matrix sequence P_k, $k = 0, 1, \cdots, N$ such that

$$(A_k + B_k K_k)^T P_{k+1}(A_k + B_k K_k) - P_k + K_k^T R K_k + Q < 0, \; P_N = S. \tag{2.3.4}$$

Lemma 2.3.1 *[XS95] Consider the system (2.3.1) with the cost function (2.3.3) and suppose the control law $u(k) = K_k x(k)$ is a GCC associated with a cost matrix P_0. Then the closed-loop system achieves $J < x^T(0)P_0x(0)$.*

2.3.1 State Feedback

From Definition 2.3.1 and Lemma 2.3.1, we know that if there exists a state feedback control $u(k) = K_k x(k)$ such that (2.3.4) is satisfied, then it is a guaranteed cost control and the cost function satisfies $J < x^T(0)P_0x(0)$. From the Schur complement, it is easy to show that the inequality (2.3.4) is equivalent to

$$\begin{bmatrix} -P_k & A_k^T + K_k^T B_k^T & Q^{1/2} & K_k^T R^{1/2} \\ A_k + B_k K_k & -P_{k+1}^{-1} & 0 & 0 \\ Q^{1/2} & 0 & -I & 0 \\ R^{1/2}K_k & 0 & 0 & -I \end{bmatrix} < 0, \;\; k = 0, 1, \cdots, N-1, \;\; P_N = S. \tag{2.3.5}$$

Clearly, the above inequality is a DLMI. Given P_N, we can recursively solve it until a P_0 is obtained. So we have the following theorem.

Theorem 2.3.1 *Consider the system (2.3.1) and the cost function (2.3.3). Suppose there exists a K_k such that the DLMI (2.3.5) is satisfied, then a control law given by $u(k) = K_k x(k)$ will achieve $J < x^T(0)P_0x(0)$.*

From the discussion above, we can see that the DLMI is solved linearly in matrix P_k at each step. This linearity can give a simple and easily realizable method to deal with polytopic uncertainty in both the finite horizon and the infinite case. To state this, we will first consider an alternative problem in the following. Representing a finite set of plants involved by $\Phi \triangleq \{\mathcal{S}_i, i = 1, 2, \cdots, M_p\}$, where the system $\mathcal{S}_i$ is described by

$$x(k+1) \;\; = \;\; A_k^{(i)} x(k) + B_k^{(i)} u(k), \;\; k = 0, 1, \cdots, N-1 \tag{2.3.6}$$

$$y(k) = C_k^{(i)} x(k) \tag{2.3.7}$$

We look for a single guaranteed cost control of the form $u(k) = K_k x(k)$, $k = 0, \cdots, N-1$, which ensures that $J < x^T(0)P_0x(0)$. To this end, we have the following theorem.

Theorem 2.3.2 *Consider the set of plants (2.3.6) and the cost function (2.3.3). Suppose there exists a* K_k *such that*

$$\begin{bmatrix} -P_k^{(i)} & A_k^{(i)T} + K_k^T B_k^{(i)T} & Q^{1/2} & K_k^T R^{1/2} \\ A_k^{(i)} + B_k^{(i)} K_k & -(P_{k+1}^{(i)})^{-1} & 0 & 0 \\ Q^{1/2} & 0 & -I & 0 \\ R^{1/2} K_k & 0 & 0 & -I \end{bmatrix} < 0, \; P_N = S \tag{2.3.8}$$

for $i = 0, 1, \cdots, M_p$ *and* $k = 0, 1, \cdots, N-1$, *then a control law given by* $u(k) = K_k x(k)$ *will achieve* $J < x^T(0)P_0x(0)$ *for the set of plants.*

Remark 2.3.1 In Theorem 2.3.1, if we minimize $trace(P_k)$ subject to (2.3.5), we can achieve the minimum cost bound. Similarly, in Theorem 2.3.2, we can minimize $trace(P_k^{(i)})$ subject to (2.3.8) to obtain the minimum guaranteed cost bound.

Remark 2.3.2 For the case of uncertain systems with polytopic uncertainties with vertices at $\{A_k^{(i)}, B_k^{(i)}, C_k^{(i)}\}$, since the method above is not convex, the actual cost for uncertainty inside the polytope can't be guaranteed to be less than the bound obtained by Theorem 2.3.2. Thus the DLMI method can't be applied directly to polytopic uncertainty. Due to the lack of convexity, it is not sufficient to only consider the vertices for systems with polytopic uncertainty. However we can resort to the grid method in [GPS] to grid the polytope into finite set of plants, which can be made dense enough to cover the former. The validity of the combination of the DLMI and the grid methods will be demonstrated in the following section.

2.3.2 Static Output Feedback

In this subsection we want to design a guaranteed cost control by using static output feedback.

Theorem 2.3.3 *Consider the system (2.3.1)-(2.3.2) and the cost function (2.3.3). A control* $u(k) = K_k y(k)$ *is said to be a guaranteed cost control if it satisfies the following DMLI*

$$\begin{bmatrix} -P_k & A_k^T + C^T K_k^T B_k^T & Q^{1/2} & K_k^T R^{1/2} \\ A_k + B_k K_k C_k & -P_{k+1}^{-1} & 0 & 0 \\ Q^{1/2} & 0 & -I & 0 \\ R^{1/2} K_k & 0 & 0 & -I \end{bmatrix} < 0, \;\; P_N = S \qquad (2.3.9)$$

for $k = 0, 1, \cdots, N-1$*. And under the control law, the cost function is less than* $x^T(0) P_0 x(0)$*.*

Similarly, for the system (2.3.6) we have the following theorem to deal with the guaranteed cost control problem for a finite set of plants.

Theorem 2.3.4 *Consider the set of plants (2.3.6)-(2.3.7) and the cost function (2.3.3). Suppose there exists a* K_k *such that*

$$\begin{bmatrix} -P_k^{(i)} & A_k^{(i)T} + C_k^{(i)T} K_k^T B_k^{(i)T} & Q^{1/2} & K_k^T R^{1/2} \\ A_k^{(i)} + B_k^{(i)} K_k C_k^{(i)} & -(P_{k+1}^{(i)})^{-1} & 0 & 0 \\ Q^{1/2} & 0 & -I & 0 \\ R^{1/2} K_k & 0 & 0 & -I \end{bmatrix} < 0, \quad P_N = S \qquad (2.3.10)$$

for $i = 0, 1, \cdots, M_p$ *and* $k = 0, 1, \cdots, N-1$*, then a control law given by* $u(k) = K_k y(k)$ *will achieve* $J < x^T(0) P_0 x(0)$ *for the set of plants.*

Remark 2.3.3 The attractive advantage of the DLMI method of Theorem 2.3.4 is that the inequality (2.3.10) is linear in $P_k^{(i)}$ and K_k respectively at each iteration step. To deal with the GCC problem for systems with polytopic uncertainties, we can use the method introduced in Remark 2.3.2.

2.4 Illustrative Examples

2.4.1 Infinite Horizon Case

The system to be controlled is obtained from the discretized model of an inverted pendulum on a cart [XS95, Gar97]:

$$x(k+1) = \begin{bmatrix} 1.05 & 0.11-0.08\delta & 0.0 & 0.0-0.08\delta \\ 1.10 & 1.05-0.8\delta & 0.0 & 0.0-0.8\delta \\ -0.02 & -0.01+0.08\delta & 1.0 & 0.1+0.08\delta \\ -0.1 & -0.01+0.8\delta & 0.0 & 1.0+0.8\delta \end{bmatrix} x(k) + \begin{bmatrix} -0.05 \\ -1.0 \\ 0.05 \\ 0.0 \end{bmatrix} u(k) \tag{2.4.1}$$

where $x(k)$ is the state and δ is an unknown parameter satisfying $|\delta| \le \delta_0$. The cost function is given by (2.2.11) with $Q = diag\{2, 2, 0.5, 0.5\}$ and $R = 0.01$. Here we assume all the states are available for feedback and $\mathcal{E}(x(0)x^T(0)) = I$.

It is required to place the poles of the uncertain closed-loop system in a given disc. By solving the optimization problem in Remark 2.2.3 for different center and radius combinations under different uncertainty bounds δ_0, the guaranteed cost bounds are given in Table 2.1.

From the table we can see clearly that the larger the δ_0 the larger the guaranteed cost bound. Also the guaranteed cost bound increases when the center of the given disc in which the poles to be placed goes much far away from the origin. This can be explained by the fact that when the poles of a system are much away from the origin its transient response is much worse and the guaranteed cost bound is higher.

	$\eta = 0.0$ $\rho = 1.0$	$\eta = 0.1$ $\rho = 0.9$	$\eta = 0.2$ $\rho = 0.8$	$\eta = 0.3$ $\rho = 0.7$	$\eta = 0.4$ $\rho = 0.6$	$\eta = 0.5$ $\rho = 0.5$
$\delta_0 = 0.0$	210.2351	223.9954	242.6363	267.4758	304.4669	363.2661
$\delta_0 = 0.3$	236.6869	255.8579	283.6387	327.2695	404.5191	570.9921
$\delta_0 = 0.5$	264.2008	292.6217	336.7754	412.3196	562.1930	956.8210
$\delta_0 = 0.7$	299.8696	341.7581	409.7693	533.1205	804.4985	1688.700
$\delta_0 = 1.0$	367.7991	437.1782	555.9190	791.2459	1397.600	4333.900
$\delta_0 = 3.0$	1319.000	2014.300	3800.000	11666.00	634940.0	/
$\delta_0 = 5.0$	4111.400	8886.500	42153.00	/	/	/
$\delta_0 = 7.0$	15039.00	17850.00	/	/	/	/

Table 2.1 Guaranteed cost bounds for different cases.

When it is required to place the closed-loop poles in a disc with $\eta = 0.5$ and $\rho = 0.5$, for $\delta_0 = 0.5$ the guaranteed cost bound is 956.8210 which is significantly less than the result in [Gar97] where the corresponding bound is 1957.0. So the method in this chapter is less conservative than that in [Gar97]. For this case, the guaranteed cost control gain is given by

$$K_1 = [\,5.1668 \quad 1.7448 \quad 0.1355 \quad 0.6858\,]$$

under which the poles of the uncertain closed-loop system subjected to the parameter δ are shown in Figure 2.1.1. When $\delta = 0.5$, the poles are

$$\lambda(A_c) = \{0.7532 + 0.3505i, \quad 0.7532 - 0.3505i, \quad 0.2944, \quad 0.9888\}$$

and the state trajectories are shown in Figure 2.2.

When only the guaranteed cost control problem without the pole-placement constraint is considered, i.e. $\eta = 0.0$ and $\rho = 1.0$, the guaranteed cost control gain for $\delta_0 = 1.0$ is

$$K_2 = [\,8.6892 \quad 3.0761 \quad 0.4112 \quad 1.6167\,]$$

and the guaranteed cost control bound is 367.7991. Under the feedback gain and

	$\eta = 0.0$ $\rho = 1.0$	$\eta = 0.1$ $\rho = 0.9$	$\eta = 0.2$ $\rho = 0.8$	$\eta = 0.3$ $\rho = 0.7$	$\eta = 0.4$ $\rho = 0.6$	$\eta = 0.5$ $\rho = 0.5$
$Theorem 2.2.2$	264.2008	292.6217	336.7754	412.3196	562.1930	956.8210
$Algorithm$	262.1446	291.9157	335.3586	410.1075	557.9459	941.5108

Table 2.2 The comparison between Theorem 2.2.2 and the iterative algorithm.

δ varying from -1 to +1, the poles of the closed-loop uncertain system are shown in Figure 2.1.2. When $\delta_0 = 0.5$ the bound is 264.2008 which is less than 266.3 in [XS95]. This also demonstrates the less conservatism and effectiveness of our method. For this case, the poles are

$$\lambda(A_c) = \{0.7140 + 0.2942i, \quad 0.7140 - 0.2942i, \quad -0.0235, \quad 0.9734\}$$

and the state trajectories can also be found in Figure 2.2 which shows the difference between the cases of $\eta = 0.0$, $\rho = 1.0$ and $\eta = 0.5$, $\rho = 0.5$. From the different responses, it is clear that the locations of the poles of the closed-loop system can also affect the guaranteed cost bound.

To show the effectiveness of the proposed iterative algorithm, we consider the case of $\delta_0 = 0.5$. Combining the algorithm and Theorem 2.2.2, the corresponding results are shown in Table 2.2.

From the results given above, we can see clearly that the conservativeness can be further reduced by the proposed iterative algorithm. When $\eta = 0.5$ and $\rho = 0.5$, the feedback gain converges to

$$K = [\,5.1764 \quad 1.7505 \quad 0.1385 \quad 0.6877\,]$$

after 6 iterations if the error tolerance between the k-th and $(k-1)$-th cost bounds is set to 0.04. The obtained feedback K above is very close to that obtained by Theorem 2.2.2 only. A comparison among the method of [XS95], Theorem 2.2.3 and the iterative algorithm is shown in Figure 2.3.1. A comparison among the method of [Gar97], Theorem 2.2.2 and the iterative algorithm is shown in Figure 2.3.2.

2.4.2 Finite Horizon Case

First we apply the DLMI method to the nominal system without uncertainties to illustrate its compliance with the results obtained by Theorem 2.2.3. For the system (2.4.1), when $\delta = 0$, the feedback gain

$$K_3 = [\,5.6752 \quad 1.9587 \quad 0.3534 \quad 0.6792\,]$$

is obtained by using Theorem 2.2.3. For the finite horizon case, we let $N = 100$ and consider two cases of $S_{N_1} = diag\{5, 5, 5, 5\}$ and $S_{N_2} = diag\{20, 20, 20, 20\}$. By using Theorem 2.3.1, for both the cases of S_{N_1} and S_{N_2} we can obtain a stationary controller

$$K_4 = [\,5.6715 \quad 1.9577 \quad 0.3529 \quad 0.6784\,],$$

which is very close to K_3. And the guaranteed cost bound is 210.8581. The convergence of the guaranteed cost bound and the feedback gain is shown in Figure 2.4, from which we can see that when there is no system uncertainty the controller converges to a stationary one which is almost the same as that obtained by Theorem 2.2.3.

When taking into account the polytopic uncertainty, we will show that less conservative results can be obtained by using the DLMI method. It should be noted first that when minimizing different weighted sums of the traces of the solutions of DLMIs different feedback gains and guaranteed cost bounds will be obtained. The corresponding results are shown in Table 2.3, where a:b means the proportion of weights between the trace of DLMI solution at $\delta = 0.5$ and that at $\delta = -0.5$.

From Table 2.3 we can see that by the DLMI method the lowest bound of 262.6555

Weighted sums	0.15 : 0.85	0.25 : 0.75	0.35 : 0.65	0.38 : 0.62	0.45 : 0.55
Cost bound	289.9680	276.4387	265.2801	262.6555	275.1254
Weighted sums	0.5 : 0.5	0.55 : 0.45	0.65 : 0.35	0.75 : 0.25	0.85 : 0.15
Cost bound	287.1077	302.6509	347.7592	431.3045	629.6661

Table 2.3 Guaranteed cost bounds for different weighted sums of the traces of DLMIs' solutions in finite horizon by state feedback.

is obtained in the case of 0.38:0.62 and the feedback gain is

$$K_5 = [\,6.3234 \quad 2.3512 \quad 0.3410 \quad 1.0377\,]\,.$$

Although this guaranteed cost bound is very close to that obtained by the iterative algorithm, the actual bound under K_5 is much lower than that under

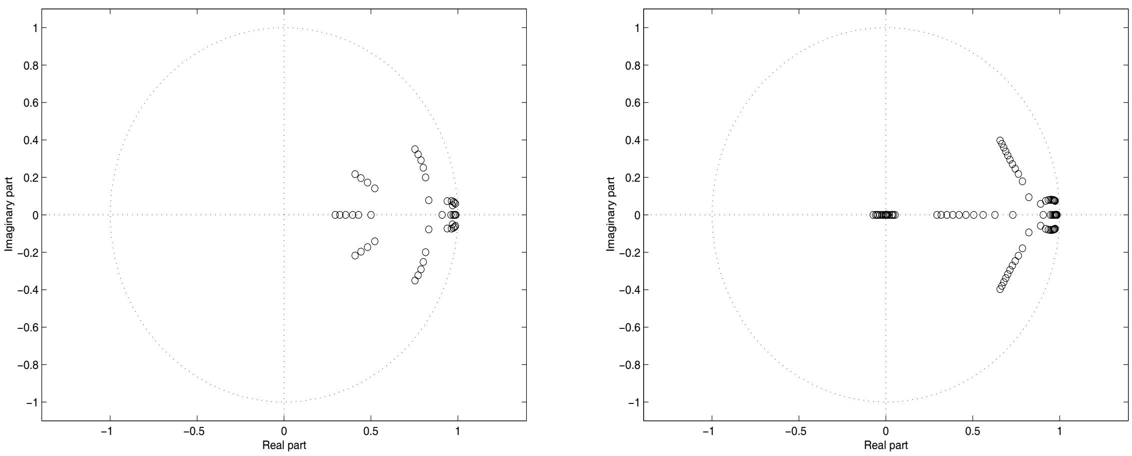
$$K_6 = [\,7.0710 \quad 2.4657 \quad 0.4186 \quad 1.1153\,]$$

which is obtained by the iterative algorithm, especially when δ varies from -0.3 to 0.1. This is clearly shown in Figure 2.5.1 which demonstrates the less conservatism of the DLMI method than Theorem 2.2.3 and the iterative algorithm.

The actual cost bound related to the weighted sums from 0.15:0.85 to 0.55:0.45 are shown in Figure 2.5.2, from which it can be seen that for each case the actual cost is less than the corresponding guaranteed cost bound for the 11 systems. To further test the validity of the DLMI method we vary δ from -0.5 to 0.5 by an increment of 0.001 and we find that for each case, the actual cost is still less than the corresponding guaranteed cost bound for all these 1001 systems. The convergence curves are shown in Figure 2.6 at the vertex of $\delta = -0.5$ and $\delta = 0.5$ for both the cases of $S_N = diag\{5, 5, 5, 5\}$ and $S_N = diag\{20, 20, 20, 20\}$.

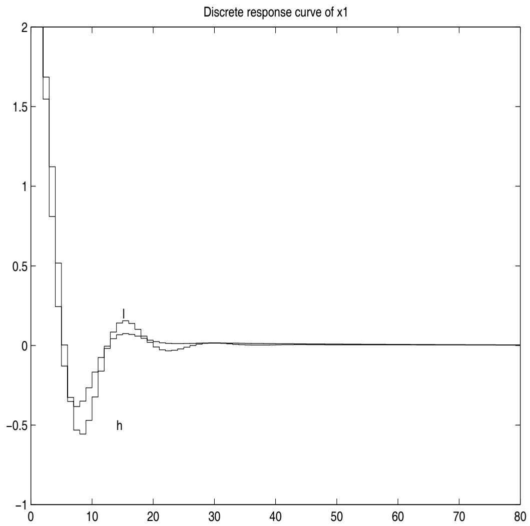
2.5 Conclusion

This chapter has focused on the problem of robust guaranteed cost control for discrete-time uncertain systems. In the infinite horizon case, LMI conditions were developed for GCC with pre-specified disc pole location in the presence of polytopic uncertainty. These conditions are derived based on a parameter-dependent Lyapunov function approach. Thus the conservatism is reduced greatly. The convexity of the problem ensures that a global optimum is reachable when it exists. An iterative algorithm was also proposed to reduce the conservatism further. When the pole-placement constraint is not considered, we have developed a DLMI method to tackle the GCC problem over the finite horizon for linear time-varying systems. The examples show that this method can give the least conservative results than the former two methods.

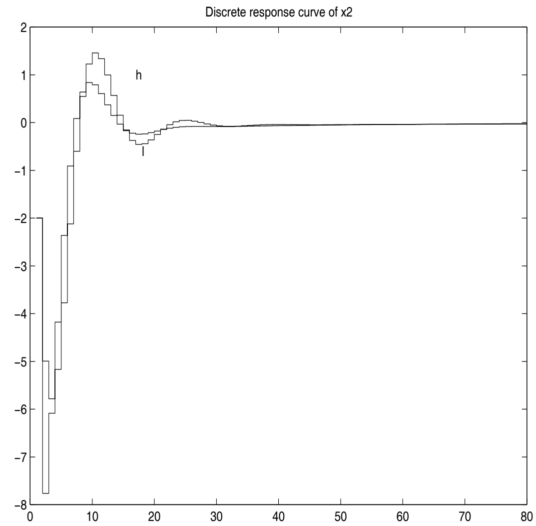


2.1.1 $\eta = 0.5$, $\rho = 0.5$ and $r \leq 0.5$ 2.1.2 $\eta = 0.0$, $\rho = 1.0$ and $r \leq 1.0$

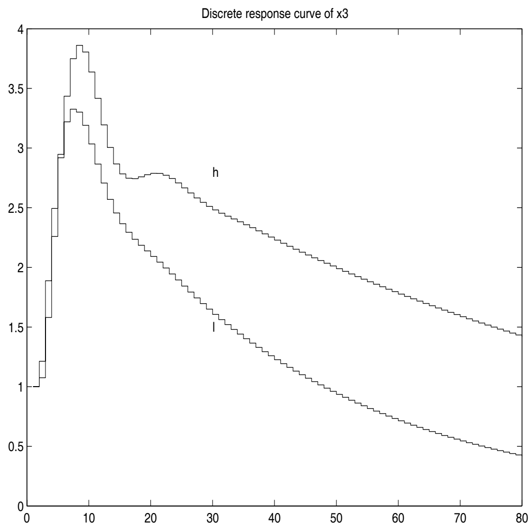
Figure 2.1 Poles of the closed-loop uncertain system.



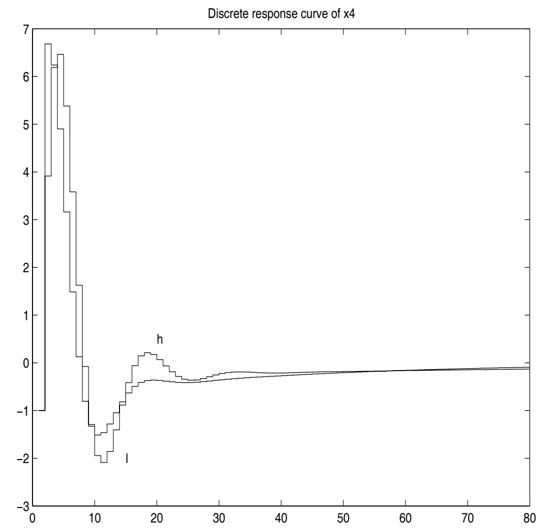
2.2.1 Trajectory of x_1



2.2.2 Trajectory of x_2

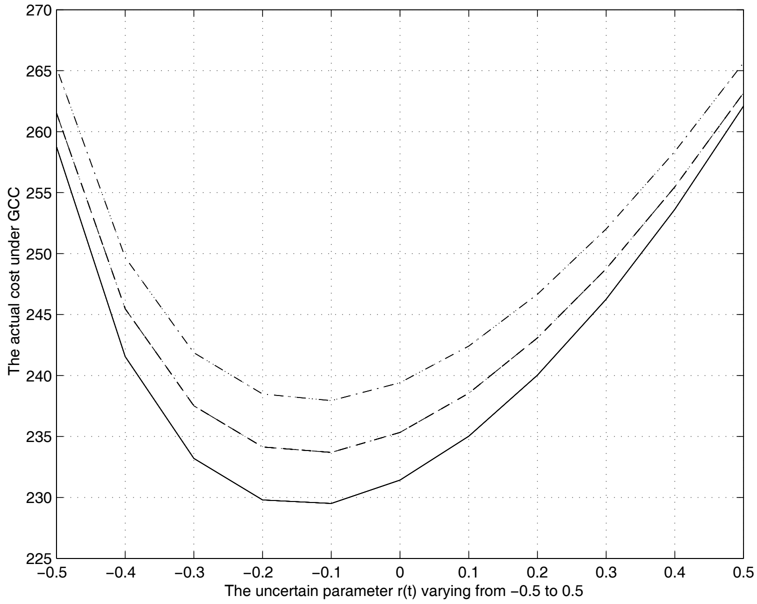


2.2.3 Trajectory of x_3

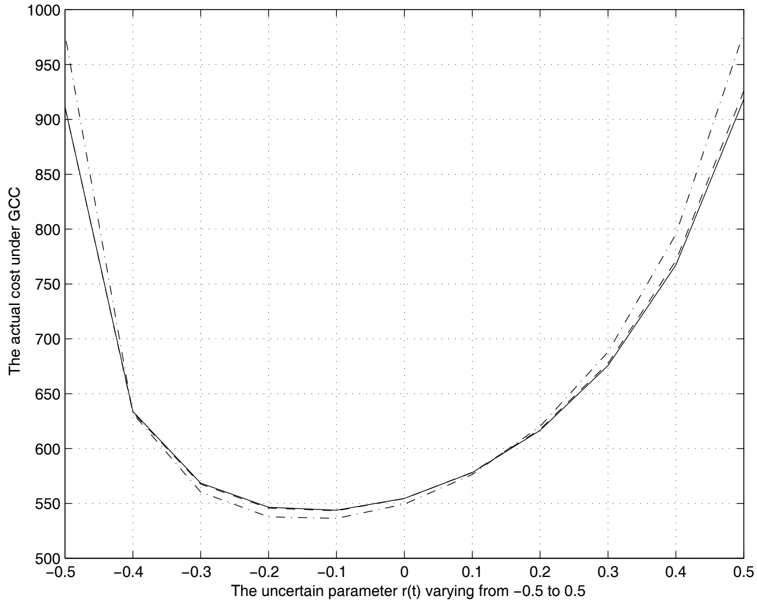


2.2.4 Trajectory of x_4

Figure 2.2 Response curves of state x, where h refers to the case of $\eta = 0.5,\ \rho = 0.5$ and l refers to the case of $\eta = 0.0,\ \rho = 1.0$.

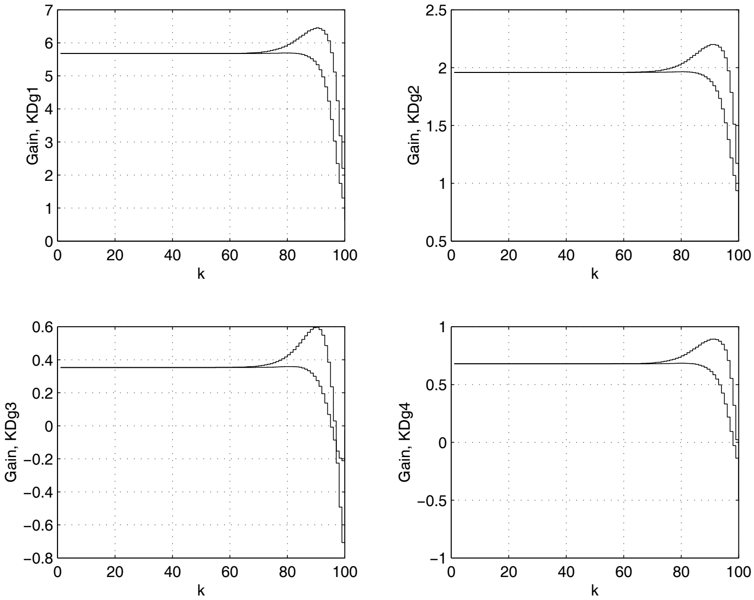


2.3.1 The case of $\eta = 0.0$, $\rho = 1.0$. The dashdot is for [XS95], the dashed is for Theorem 2.2.3 and the solid is for the proposed iterative algorithm

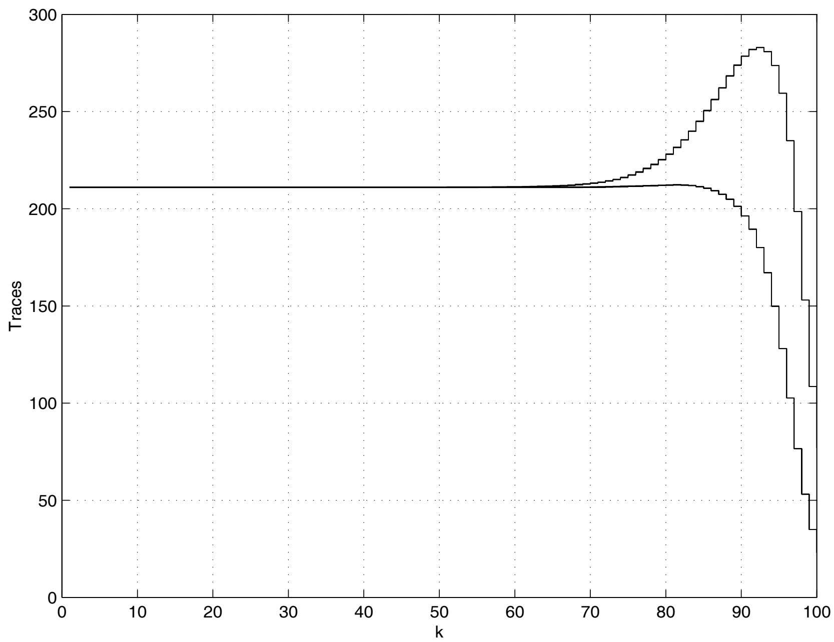


2.3.2 The case of $\eta = 0.5$, $\rho = 0.5$. The dashed is for Theorem 2.2.2, the dashdot is for [Gar97] and the solid is for the proposed iterative algorithm

Figure 2.3 Actual cost bound versus uncertain parameter under GCC (infinite horizon case).

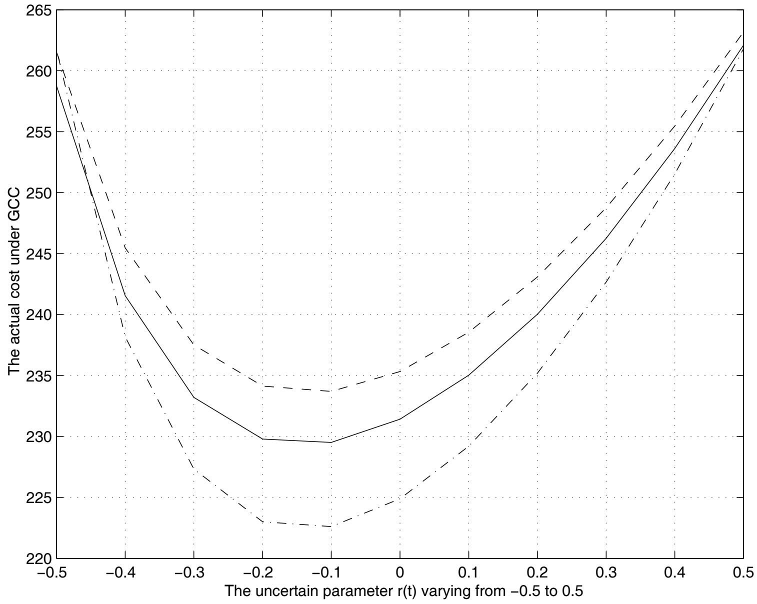


2.4.1 The convergence of the feedback gain of GCC for the cases of S_{N1} and S_{N2}

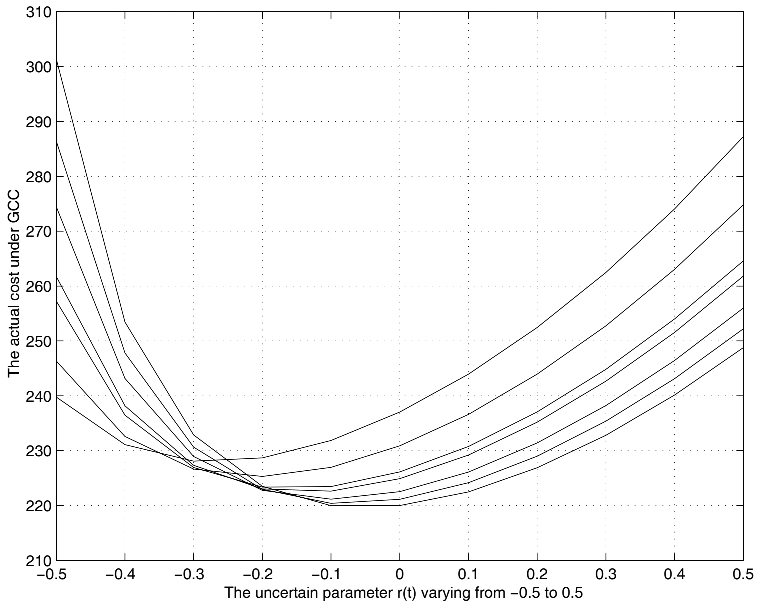


2.4.2 The convergence of the traces of the solutions of DLMIs for the cases of S_{N1} and S_{N2}

Figure 2.4 The convergence of the DLMI method for the nominal system.

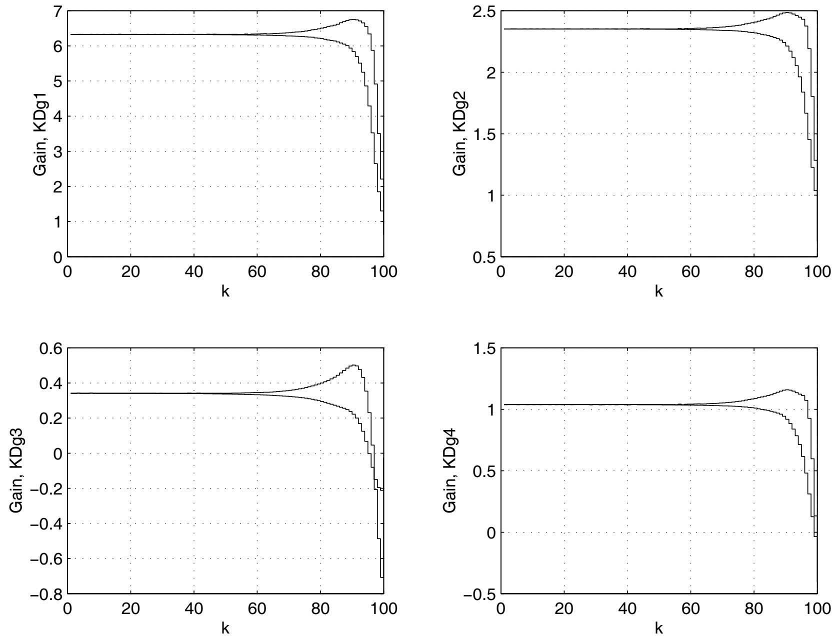


2.5.1 The dashed is for Theorem 2.2.2, the solid is for the proposed iterative algorithm and the dashdot is for DLMI method in finite horizon

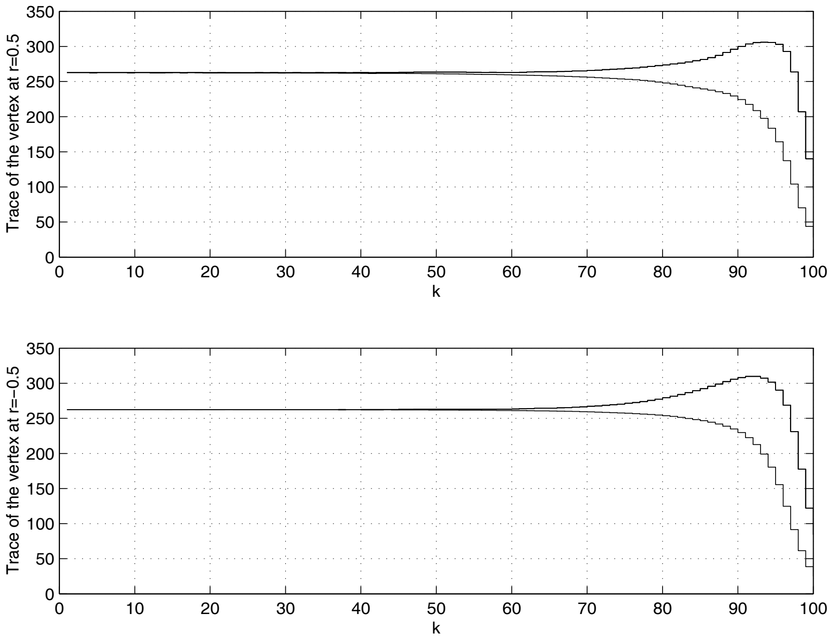


2.5.2 Actual cost bound versus uncertain parameter under GCC for different weighted sums of the traces of the DLMIs' solutions

Figure 2.5 Actual cost bound versus uncertain parameter under GCC (finite horizon case).



2.6.1 The convergence of the feedback gain of GCC for the cases of S_{N1} and S_{N2}



2.6.2 The convergence of the traces of the solutions of DLMIs for the cases of S_{N1} and S_{N2}

Figure 2.6 The convergence of the DLMI method for the uncertain system.

Chapter 3

Robust H_2 and H_∞ Control of Systems with Polytopic Uncertainties via Dynamic Output Feedback

3.1 Introduction

Uncertain systems with polytopic uncertainties have received much interest in recent years because the polytopic characterization of uncertainty can lead to less conservative results than that of the norm-bounded one. The H_2 and H_∞ filtering problems for discrete-time systems with polytopic uncertainties have been solved successfully in [GdO98, Ger99, GBGdO00]. And less conservative results are presented in [SXS01, GdOB02]. The state feedback and static output feedback control problems for such systems have also been investigated in [dOGB02, Sha02]. However, the dynamic output feedback control problem for discrete-time systems with polytopic uncertainties has not been solved because of the difficulties in obtaining fixed controller parameters from the solutions of LMIs.

In this chapter, we will develop a method to convexify the output feedback control design problem for discrete-time systems with polytopic uncertainties. Based on the technique of change of variables, the H_2 and H_∞ controller parameters can be obtained in terms of LMIs.

3.2 Problem Formulation

Consider the following system:

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \tag{3.2.1}$$

$$z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k) \tag{3.2.2}$$

$$y(k) = C_2 x(k) + D_{21} w(k) \tag{3.2.3}$$

where $x(k) \in \mathcal{R}^n$ is the state vector, $w(k) \in \mathcal{R}^p$ is the disturbance input, $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^r$ is the output, and $z(k) \in \mathcal{R}^q$ is the controlled output.

The matrices $A,\ B_1,\ B_2,\ C_1,\ C_2,\ D_{11},\ D_{12}$ and D_{21} are appropriately dimensioned. They belong to the following uncertainty polytope:

$$\begin{aligned}\Omega\ &= \{(A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}) \mid (A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}) = \\ &\sum_{i=1}^{M} \theta_i (A^{(i)}, B_1^{(i)}, B_2^{(i)}, C_1^{(i)}, C_2^{(i)}, D_{11}^{(i)}, D_{12}^{(i)}, D_{21}^{(i)}), \theta_i \geq 0,\ \sum_{i=1}^{M} \theta_i = 1 \Big\}\end{aligned} \tag{3.2.4}$$

For the convenience of expression, it is assumed that $C_2^{(i)} = C_2$ and $D_{21}^{(i)} = D_{21}$ for $i = 1, 2, \cdots, M$. Without loss of generality, we shall also assume $p = q$, i.e., the disturbance input and the signal to be estimated have the same dimension. Note that, if this is not the case, some simple modification can render the requirement being satisfied. For example, if $p < q$, the matrices B_1, D_{21} and D_{11} can be extended as $B_1' = [B_1 \;\; 0_{n\times(q-p)}]$, $D_{21}' = [D_{21} \;\; 0_{q\times(q-p)}]$ and $D_{11}' = [D_{11} \;\; 0_{n\times(q-p)}]$.

Let a controller for the system (3.2.1)-(3.2.3) be of the form:

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}y(k) \qquad (3.2.5)$$

$$u(k) = \hat{C}\hat{x}(k) + \hat{D}y(k) \qquad (3.2.6)$$

where $\hat{x}(k) \in \mathcal{R}^{\hat{n}}$ is the state of the controller, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ are the controller matrices to be determined.

First, denote $\xi(k) = [\,x^T(k) \quad \hat{x}^T(k)\,]^T$. It follows from (3.2.1)-(3.2.3) and (3.2.5)-(3.2.6) that

$$\xi(k+1) = \bar{A}\xi(k) + \bar{B}w(k) \qquad (3.2.7)$$

$$z(k) = \bar{C}\xi(k) + \bar{D}w(k) \qquad (3.2.8)$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix}, \; \bar{B} = \begin{bmatrix} B_2\hat{D}D_{21} + B_1 \\ \hat{B}D_{21} \end{bmatrix} \\ \bar{C} &= [\,C_1 + D_{12}\hat{D}C_2 \quad D_{12}\hat{C}\,], \quad \bar{D} = D_{11} + D_{12}\hat{D}D_{21}. \end{aligned}$$

The H_2 or H_∞ control problem in this chapter can be stated as follows: find a dynamic output feedback controller of the form of (3.2.5)-(3.2.6) such that the closed-loop system (3.2.7)-(3.2.8) is asymptotically stable and has an optimal upper bound for H_2 or H_∞ performance for all uncertainties belonging to the polytope Ω.

Remark 3.2.1 The dynamic output feedback control of uncertain systems with the norm-bounded type of uncertainty has been successfully solved via a combination of the LMI approach and scaling parameters searching; see [EP00] and the references therein. However, the dynamic output feedback H_2 or H_∞ control of discrete-time systems with polytopic uncertainty has not been addressed in existing literature because of the difficulty in convexification.

3.3 Robust H_2 Controller Design

To find a robust H_2 dynamic output feedback controller for the system with polytopic uncertainties, we first give the following useful technical lemma.

Lemma 3.3.1 *Let $\Delta \in \mathcal{R}^{n\times n} > 0$, $\Psi \in \mathcal{R}^{p\times p} > 0$, $W_2 \in \mathcal{R}^{n\times n}$, $W_1 \in \mathcal{R}^{n\times n}$, $\Xi \in \mathcal{R}^{n\times n}$, $\Omega_1 \in \mathcal{R}^{p\times n}$ and $\Omega_2 \in \mathcal{R}^{p\times n}$ with W_1 being nonsingular. There exists a matrix $H > 0$ such that*

$$\begin{bmatrix} W_1^T H W_1 & \Xi^T & \Omega_1^T \\ \Xi & \Delta - W_2^T H W_2 & \Omega_2^T \\ \Omega_1 & \Omega_2 & \Psi \end{bmatrix} > 0 \tag{3.3.1}$$

if there exist matrices $\bar{H} > 0$, $\hat{H} > 0$ and $\tilde{H}$ satisfying $\bar{H}\tilde{H} = I$ such that

$$\begin{bmatrix} \bar{H} & \Xi^T & \Omega_1^T \\ \Xi & \Delta - \hat{H} & \Omega_2^T \\ \Omega_1 & \Omega_2 & \Psi \end{bmatrix} > 0 \tag{3.3.2}$$

and

$$\begin{bmatrix} \tilde{H} & W_1^{-1} W_2 \\ W_2^T W_1^{-T} & \hat{H} \end{bmatrix} > 0 \tag{3.3.3}$$

Further, (3.3.3) will be implied if for some scalar $\varepsilon > 0$, the following holds:

$$\begin{bmatrix} 2\varepsilon I - \varepsilon^2 \bar{H} & W_1^{-1} W_2 \\ W_2^T W_1^{-T} & \hat{H} \end{bmatrix} > 0. \tag{3.3.4}$$

Proof Letting $\hat{H} = W_2^T H W_2$, (3.3.2) implies (3.3.1) if

$$\bar{H} < W_1^T H W_1 = W_1^T W_2^{-T} \hat{H} W_2^{-1} W_1. \tag{3.3.5}$$

It can be seen easily that (3.3.5) holds if and only if $\bar{H}^{-1} > W_1^{-1} W_2 \hat{H}^{-1} W_2^T W_1^{-T}$, which by Schur complement leads to (3.3.3).

Further, since $(\bar{H}^{-1} - \varepsilon I)^T \bar{H} (\bar{H}^{-1} - \varepsilon I) \geq 0$, we have $\bar{H}^{-1} \geq 2\varepsilon I - \varepsilon^2 \bar{H}$. Therefore, we know that (3.3.5) holds if

$$2\varepsilon I - \varepsilon^2 \bar{H} > W_1^{-1} W_2 \hat{H}^{-1} W_2^T W_1^{-T},$$

which is equivalent to (3.3.4).

In the case when the system (3.2.1)-(3.2.3) has no parameter uncertainty and given a controller that stabilizes the system, the H_2 norm square of the closed-loop system (3.2.7)-(3.2.8) can be obtained by solving the following optimization:

$$\min_{(Q^T=Q, S^T=S)} trace(S) \tag{3.3.6}$$

subject to

$$\bar{A}^T Q \bar{A} - Q + \bar{C}^T \bar{C} < 0 \tag{3.3.7}$$

$$\bar{B}^T Q \bar{B} + \bar{D}^T \bar{D} < S \tag{3.3.8}$$

An alternative characterization of the H_2 norm of a discrete-time LTI system (3.2.7)-(3.2.8) without parameter uncertainty is given in [DXTG03], which is recalled below.

Lemma 3.3.2 *[DXTG03] The H_2 norm square of the system (3.2.7)-(3.2.8) can be obtained by the following minimization:*

$$\min_{(Q^T=Q, S^T=S, \Sigma)} trace(S) \tag{3.3.9}$$

subject to

$$\tilde{A}^T diag\{Q, I\}\tilde{A} - \begin{bmatrix} Q & \Sigma \\ \Sigma^T & S \end{bmatrix} < 0 \tag{3.3.10}$$

where

$$\tilde{A} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}.$$

Proof First, (3.3.10) can be rewritten as

$$\begin{bmatrix} \bar{A}^T Q\bar{A} - Q + \bar{C}^T\bar{C} & \bar{A}^T Q\bar{B} + \bar{C}^T\bar{D} - \Sigma \\ \bar{B}^T Q\bar{A} + \bar{D}^T\bar{C} - \Sigma^T & \bar{B}^T Q\bar{B} + \bar{D}^T\bar{D} - S \end{bmatrix} < 0. \tag{3.3.11}$$

It is then clear from (3.3.11) that if there exists a solution (Q, S, Σ) to (3.3.10), the (1,1)-th block and (2,2)-th block of (3.3.10) imply (3.3.7) and (3.3.8), respectively. On the other hand, if there exists a solution (Q, S) satisfying (3.3.7) and (3.3.8), then by letting $\Sigma = \bar{A}^T Q\bar{B} + \bar{C}^T\bar{D}$, (3.3.10) is also satisfied with the same Q and S.

Remark 3.3.1 The characterization of the H_2 norm in the above lemma has the advantage that the parameter Σ in (3.3.10) will give an additional freedom when designing controllers for uncertain systems. Furthermore, it provides a unified treatment of H_2 and H_∞ control via the LMI approach. In fact, the H_∞ norm of the closed-loop system (3.2.7)-(3.2.8) is less than γ if and only if (3.3.10) has a positive definite solution for $\Sigma = 0$ and $S = \gamma^2 I$ [Xie96] .

It is easy to know [dOGH99] that (3.3.10) holds for some matrices Q, S and Σ if and only if there exist matrices Q, S, Σ and Φ such that

$$\begin{bmatrix} -Q & -\Sigma & \bar{A}^T\Phi & \bar{C}^T \\ -\Sigma^T & -S & \bar{B}^T\Phi & \bar{D}^T \\ \Phi^T\bar{A} & \Phi^T\bar{B} & Q - (\Phi + \Phi^T) & 0 \\ \bar{C} & \bar{D} & 0 & -I \end{bmatrix} < 0. \tag{3.3.12}$$

Remark 3.3.2 (3.3.10) and (3.3.12) are equivalent in computing the H_2 norm for the system without uncertainty. However for uncertain systems, (3.3.12) will give a less conservative design due to the additional freedom in optimization given by the matrix variables Φ and Σ.

An upper bound of the H_2 norm square of the uncertain closed-loop system (3.2.7)-(3.2.8) can be computed by

$$\min_{(Q^T=Q, S^T=S, \Sigma)} trace(S) \tag{3.3.13}$$

subject to

$$\begin{bmatrix} -Q^{(i)} & -\Sigma^{(i)} & \bar{A}^{(i)T}\Phi & \bar{C}^{(i)T} \\ -\Sigma^{(i)T} & -S & \bar{B}^{(i)T}\Phi & \bar{D}^{(i)T} \\ \Phi^T\bar{A}^{(i)} & \Phi^T\bar{B}^{(i)} & Q^{(i)} - (\Phi + \Phi^T) & 0 \\ \bar{C}^{(i)} & \bar{D}^{(i)} & 0 & -I \end{bmatrix} < 0. \tag{3.3.14}$$

Since Φ is invertible due to $\Phi + \Phi^T > Q^{(i)} > 0$, we denote

$$\Phi = \begin{bmatrix} X & \bar{M} \\ M_1 & U \end{bmatrix}, \ \Phi^{-1} = \begin{bmatrix} Y & \bar{N} \\ N_1 & V \end{bmatrix}$$

and

$$J = \begin{bmatrix} Y & I_n \\ N_1 & 0 \end{bmatrix}, \ J_1 = diag\{J, I_n, J, I_n\}.$$

Multiplying from the left and the right of (3.3.14) by J_1^T and J_1 respectively, we obtain

$$\begin{bmatrix} -P_{11}^{(i)} & * & * \\ -P_{12}^{(i)T} & -P_{22}^{(i)} & * \\ -\Lambda_1^{(i)T} & -\Lambda_2^{(i)T} & -S \\ A^{(i)}Y + B_2^{(i)}V^{(i)} & A^{(i)} + B_2^{(i)}\hat{D}C_2 & B_1^{(i)} + B_2^{(i)}\hat{D}D_{21} \\ U^{(i)} & X^TA^{(i)} + W^{(i)}C_2 & X^TB_1^{(i)} + W^{(i)}D_{21} \\ C_1^{(i)}Y + D_{12}^{(i)}V^{(i)} & C_1^{(i)} + D_{12}^{(i)}\hat{D}C_2 & D_{11}^{(i)} + D_{12}^{(i)}\hat{D}D_{21} \end{bmatrix}$$

$$
\left.\begin{matrix}
* & * & * \\
* & * & * \\
* & * & * \\
P_{11}^{(i)} - Y^T - Y & * & * \\
P_{12}^{(i)T} - I - Z & P_{22}^{(i)} - X^T - X & * \\
0 & 0 & -I
\end{matrix}\right] < 0 \tag{3.3.15}
$$

where $P^{(i)} = [P_{lk}^{(i)}] = J_1^T Q^{(i)} J_1$, $\Lambda_j^{(i)} = J_1^T \Sigma^{(i)}$ and

$$
\begin{aligned}
U^{(i)} &= X^T(A^{(i)} + B_2^{(i)}\hat{D}C_2)Y + M_1^T\hat{B}C_2Y \\
&\quad + X^T B_2^{(i)}\hat{C}N_1 + M_1^T\hat{A}N_1 \qquad (3.3.16)\\
V^{(i)} &= \hat{C}N_1 + \hat{D}C_2Y \qquad (3.3.17)\\
W^{(i)} &= X^T B_2^{(i)}\hat{D} + M_1^T\hat{B} \qquad (3.3.18)\\
Z &= X^T Y + M_1^T N_1. \qquad (3.3.19)
\end{aligned}
$$

Although (3.3.15) is an LMI already, it is clear from (3.3.16)-(3.3.19) that one cannot obtain the fixed controller parameters $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. This imposes the major difficulty in output feedback control of systems with polytopic uncertainty. In the following, we shall address this problem by invoking Lemma 3.3.1. To this end, pre- and post-multiplying (3.3.15) by $diag\{Y^{-T}, I, I, I, X^{-T}, I\}$ and $diag\{Y^{-1}, I, I, I, X^{-1}, I\}$, respectively, and then by some row-column changes, we have

$$
\left[\begin{array}{cc|c}
-\tilde{P}_{11}^{(i)} & * & * \\
-\tilde{P}_{12}^{(i)T} & -\tilde{P}_{22}^{(i)} & * \\
\hline
A^{(i)} + B_2^{(i)}V^{(i)}Y^{-1} & A^{(i)} + B_2^{(i)}\hat{D}C_2 & \hat{P}_{11}^{(i)} - Y^T - Y \\
X^{-T}U^{(i)}Y^{-1} & A^{(i)} + X^{-T}W^{(i)}C_2 & \hat{P}_{12}^{(i)T} - X^{-T} - \bar{Z} \\
\hline
C_1^{(i)} + D_{12}^{(i)}V^{(i)}Y^{-1} & C_1^{(i)} + D_{12}^{(i)}\hat{D}C_2 & 0 \\
-\bar{\Lambda}_1^{(i)T} & -\bar{\Lambda}_2^{(i)T} & (B_1^{(i)} + B_2^{(i)}\hat{D}D_{21})^T
\end{array}\right.
$$

$$\left.\begin{array}{c|cc} * & * & * \\ * & * & * \\ \hline * & * & * \\ \hat{P}_{22}^{(i)} - X^{-T} - X^{-1} & * & * \\ \hline 0 & -I & * \\ (B_1^{(i)} + X^{-T}W^{(i)}D_{21})^T & (D_{11}^{(i)} + D_{12}^{(i)}\hat{D}D_{21})^T & -S \end{array}\right] < 0 \tag{3.3.20}$$

where $\bar{\Lambda}_1^{(i)T} = \tilde{\Lambda}_1^{(i)T} Y^{-1}$, $\bar{\Lambda}_2^{(i)} = \Lambda_2^{(i)}$, $\bar{Z} = Y + X^{-T} M_1^T N_1$ and

$$\begin{aligned} \tilde{P}^{(i)} = W_1^T P^{(i)} W_1 &= \begin{bmatrix} \tilde{P}_{11}^{(i)} & \tilde{P}_{12}^{(i)} \\ \tilde{P}_{12}^{(i)T} & \tilde{P}_{22}^{(i)} \end{bmatrix} \\ &= \begin{bmatrix} Y^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{11}^{(i)} & P_{12}^{(i)} \\ P_{12}^{(i)T} & P_{22}^{(i)} \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 \\ 0 & I \end{bmatrix}, \end{aligned} \tag{3.3.21}$$

$$\begin{aligned} \hat{P}^{(i)} = W_2^T P^{(i)} W_2 &= \begin{bmatrix} \hat{P}_{11}^{(i)} & \hat{P}_{12}^{(i)} \\ \hat{P}_{12}^{(i)T} & \hat{P}_{22}^{(i)} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & X^{-T} \end{bmatrix} \begin{bmatrix} P_{11}^{(i)} & P_{12}^{(i)} \\ P_{12}^{(i)T} & P_{22}^{(i)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X^{-1} \end{bmatrix}. \end{aligned} \tag{3.3.22}$$

From the expressions of $B_2^{(i)} V^{(i)} Y^{-1}$, $X^{-T} W^{(i)} C_2^{(i)}$, $X^{-T} U^{(i)} Y^{-1}$ in (3.3.20) and the equations (3.3.16)-(3.3.19), it is clear that the controller parameters are separated from the plant matrices. Now, we define a new set of variables as follows:

$$\bar{X} = X^{-1} \tag{3.3.23}$$

$$U = \bar{X}^T M_1^T \hat{A} N_1 Y^{-T} \tag{3.3.24}$$

$$V = \hat{C} N_1 Y^{-1} \tag{3.3.25}$$

$$W = \bar{X}^T M_1^T \hat{B}. \tag{3.3.26}$$

Then from (3.3.20) and (3.3.23)-(3.3.26), we can obtain

$$\begin{bmatrix} -\tilde{P}_{11}^{(i)} & * & * & * & * & * \\ -\tilde{P}_{12}^{(i)T} & -\tilde{P}_{22}^{(i)} & * & * & * & * \\ A^{(i)} + B_2^{(i)}V + B_2^{(i)}\hat{D}C_2 & A^{(i)} + B_2^{(i)}\hat{D}C_2 & \hat{P}_{11}^{(i)} - Y^T - Y & * & * & * \\ A^{(i)} + B_2^{(i)}\hat{D}C_2 + WC_2 + B_2^{(i)}V + U & A^{(i)} + B_2^{(i)}\hat{D}C_2 + WC_2 & \hat{P}_{12}^{(i)T} - \bar{X}^T - \bar{Z} & \hat{P}_{22}^{(i)} - \bar{X}^T - \bar{X} & * & * \\ C_1^{(i)} + D_{12}^{(i)}V + D_{12}^{(i)}\hat{D}C_2 & C_1^{(i)} + D_{12}^{(i)}\hat{D}C_2 & 0 & 0 & -I & * \\ -\bar{\Lambda}_1^{(i)T} & -\bar{\Lambda}_2^{(i)T} & (B_1^{(i)} + B_2^{(i)}\hat{D}D_{21})^T & (B_1^{(i)} + WD_{21} + B_2^{(i)}\hat{D}D_{21})^T & (D_{11}^{(i)} + D_{12}^{(i)}\hat{D}D_{21})^T & -S \end{bmatrix} < 0. \quad (3.3.27)$$

Obviously, the above matrix inequality is linear in $(U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}, S)$, however it cannot be solved directly by using the LMI approach since from (3.3.21)-(3.3.22) it is known that the variables $\tilde{P}^{(i)}$ and $\hat{P}^{(i)}$ are not independent variables. To this end, we apply Lemma 3.3.1 and have the following result.

Theorem 3.3.1 *The robust output feedback H_2 control problem for the system (3.2.1)-(3.2.3) with polytopic uncertainties is solvable if for some scalar $\varepsilon > 0$, there exists a solution $(\bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}, S)$ to the following optimization:*

$$\min trace(S)$$

subject to

$$\Psi(S, \bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}) =$$

$$\begin{bmatrix} -\bar{P}_{11}^{(i)} & * & * & * & * & * \\ -\bar{P}_{12}^{(i)T} & -\bar{P}_{22}^{(i)} & * & * & * & * \\ A^{(i)}+B_2^{(i)}V+B_2^{(i)}\hat{D}C_2 & A^{(i)}+B_2^{(i)}\hat{D}C_2 & \hat{P}_{11}^{(i)}-Y^T-Y & * & * & * \\ A^{(i)}+B_2^{(i)}\hat{D}C_2+WC_2+B_2^{(i)}V+U & A^{(i)}+B_2^{(i)}\hat{D}C_2+WC_2 & \hat{P}_{12}^{(i)T}-\bar{X}^T-\bar{Z} & \hat{P}_{22}^{(i)}-\bar{X}^T-\bar{X} & * & * \\ C_1^{(i)}+D_{12}^{(i)}V+D_{12}^{(i)}\hat{D}C_2 & C_1^{(i)}+D_{12}^{(i)}\hat{D}C_2 & 0 & 0 & -I & * \\ -\bar{\Lambda}_1^{(i)T} & -\bar{\Lambda}_2^{(i)T} & (B_1^{(i)}+B_2^{(i)}\hat{D}D_{21})^T & (B_1^{(i)}+WD_{21}+B_2^{(i)}\hat{D}D_{21})^T & (D_{11}^{(i)}+D_{12}^{(i)}\hat{D}D_{21})^T & -S \end{bmatrix} < 0 \quad (3.3.28)$$

$$\Gamma(\bar{P}_{11}^{(i)},\bar{P}_{12}^{(i)},\bar{P}_{22}^{(i)},\hat{P}_{11}^{(i)},\hat{P}_{12}^{(i)},\hat{P}_{22}^{(i)},Y,\bar{X},\varepsilon) = \begin{bmatrix} 2\varepsilon I-\varepsilon^2\bar{P}_{11}^{(i)} & * & * & * \\ -\varepsilon^2\bar{P}_{12}^{(i)T} & 2\varepsilon I-\varepsilon^2\bar{P}_{22}^{(i)} & * & * \\ Y & 0 & \hat{P}_{11}^{(i)} & * \\ 0 & \bar{X} & \hat{P}_{12}^{(i)T} & \hat{P}_{22}^{(i)} \end{bmatrix} > 0 \quad (3.3.29)$$

for $i = 1, 2, \cdots, M$. The controller parameters are given by

$$\hat{A} = M_1^{-T}\bar{X}^{-T}UYN^{-1} \quad (3.3.30)$$

$$\hat{B} = M_1^{-T}\bar{X}^{-T}W \quad (3.3.31)$$

$$\hat{C} = VYN_1^{-1} \quad (3.3.32)$$

where M_1 and N_1 satisfy $\bar{Z} = Y + \bar{X}^T M_1^T N_1$. Note that $\hat{D}$ is obtained in the optimization.

Remark 3.3.3 If for a given $\varepsilon > 0$, there exists a solution to the LMIs (3.3.28)-(3.3.29), it is easy to see that

$$\begin{bmatrix} Y^T+Y & \bar{X}+\bar{Z}^T \\ \bar{X}^T+\bar{Z} & \bar{X}^T+\bar{X} \end{bmatrix} > 0.$$

Multiplying the above from the left by $[\,I \quad -I\,]$ and from the right by $[\,I \quad -I\,]^T$, we have

$$(Y - \bar{Z}) + (Y - \bar{Z})^T < 0.$$

It is clear that $\bar{Z} - Y$ is invertible. Since $\bar{Z} = Y + X^{-T} M_1^T N_1$, M_1 and N_1 are also invertible. Thus the controller parameters in (3.3.30)-(3.3.32) can be obtained from (3.3.23)-(3.3.26).

Remark 3.3.4 Theorem 3.3.1 involves a line search for the scaling parameter ε, which can be easily carried out. On the other hand, the result may be conservative due to that a lower bound of $[\bar{P}^{(i)}]^{-1}$ is used, where

$$\bar{P}^{(i)} = \begin{bmatrix} \bar{P}_{11}^{(i)} & \bar{P}_{12}^{(i)} \\ \bar{P}_{12}^{(i)T} & \bar{P}_{22}^{(i)} \end{bmatrix}$$

To address the conservatism, we note from Lemma 3.1 that (3.3.29) can be replaced by

$$\begin{bmatrix} R_{11}^{(i)} & * & * & * \\ R_{12}^{(i)T} & R_{22}^{(i)} & * & * \\ Y & 0 & \hat{P}_{11}^{(i)} & * \\ 0 & \bar{X} & \hat{P}_{12}^{(i)T} & \hat{P}_{22}^{(i)} \end{bmatrix} > 0 \qquad (3.3.33)$$

for $i = 1, 2, \cdots, M$, where

$$R^{(i)} = \begin{bmatrix} R_{11}^{(i)} & R_{12}^{(i)} \\ R_{12}^{(i)T} & R_{22}^{(i)} \end{bmatrix} = [\bar{P}^{(i)}]^{-1} \qquad (3.3.34)$$

Note that the condition $R^{(i)}\bar{P}^{(i)} = I$ is equivalent to

$$\begin{bmatrix} \bar{P}^{(i)} & I \\ I & R^{(i)} \end{bmatrix} \geq 0 \qquad (3.3.35)$$

and $trace(\bar{P}^{(i)}R^{(i)}) = n_x$, where n_x is the dimension of $\bar{P}^{(i)}$ and $R^{(i)}$ satisfying (3.3.35). Thus, we further need to solve the following problem

$$\min \sum_{i=1}^{M} trace(\bar{P}^{(i)}R^{(i)})$$

subject to (3.3.35).

The above problem is not convex since the function $trace(P_iQ_i)$ is bilinear. This bilinear problem has been investigated by many researchers in static output control for continuous systems and many methods were proposed such as the cone complementarity linearization method by L.E. Ghaoui in [GOA97] and the sequential linear programming matrix method (SLPMM) developed by F. Leibfritz in [Lei01]. We now extend the SLPMM to solve the robust H_2 output feedback control and have the following result.

Theorem 3.3.2 *The robust optimal output feedback H_2 control problem for the system (3.2.1)-(3.2.3) with polytopic uncertainties can be solved by the following optimization:*

- *Step 1: Obtain the initial values $(\bar{P}^{(i)0}, R^{(i)0}, S^0)$ satisfying (3.3.28), (3.3.33), (3.3.35).*

- *Step 2: Given $(\bar{P}^{(i)k}, R^{(i)k})$, obtain a solution of $(\bar{P}^{(i)}, R^{(i)}, S)$, denoted by $(P_T^{(i)k}, R_T^{(i)k}, S_T^k)$, together with $(\hat{P}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, \bar{X}, Y, \bar{Z})$, to the convex optimization*

$$\min \Big(\sum_{i=1}^{M} trace(\bar{P}^{(i)}R^{(i)k} + \bar{P}^{(i)k}R^{(i)}) + trace(S) \Big)$$
$$subject\ to\ (3.3.28), (3.3.33), (3.3.35).$$

- *Step 3: If* $\left| \sum_{i=1}^{M} trace(P_T^{(i)k}R^{(i)k} + \bar{P}^{(i)k}R_T^{(i)k}) + trace(S_T^k - S^k) \right.$

$-2\sum_{i=1}^{M} trace(\bar{P}^{(i)k}R^{(i)k})\Big| \leq \epsilon$, *stop, where* ϵ *is a pre-defined sufficiently small positive scalar.*

- *Step 4: Compute* $\alpha \in [0,1]$ *by solving*

$$\min_{\alpha\in[0,1]} \sum_{i=1}^{M} \quad trace((\bar{P}^{(i)k} + \alpha(P_T^{(i)k} - \bar{P}^{(i)k})(R^{(i)k} + \alpha(R_T^{(i)k} - R^{(i)k})) \\ +trace(S^k + \alpha(S_T^k - S^k)).$$

- *Step 5: Set* $\bar{P}^{(i)(k+1)} = (1-\alpha)\bar{P}^{(i)k} + \alpha P_T^{(i)k}$, $R^{(i)(k+1)} = (1-\alpha)R^{(i)k} + \alpha R_T^{(i)k}$, $S^{k+1} = (1-\alpha)S^k + \alpha S_T^k$, *go to step 2.*

Remark 3.3.5 Suppose the above optimization leads to solutions U, V, W, $\bar{X}$, Y. Then, M_1 and N_1 and thus the controller can be computed from (3.3.30)-(3.3.32).

3.4 Robust H_∞ Controller Design

In this section, we will extend the technique in the last section to study the H_∞ control problem. Recall that when the system (3.2.1)-(3.2.3) is known, it is stable and has an H_∞ norm less than γ if there exists a matrix $P^T = P$ such that [Xie96]

$$\tilde{A}^T diag\{Q, I\}\tilde{A} - diag\{Q, \gamma^2 I\} < 0. \tag{3.4.1}$$

Note that (3.4.1) is a special case of (3.3.10) when $S = \gamma^2 I$ and $\Sigma = 0$. Thus (3.4.1) is equivalent to

$$\begin{bmatrix} -Q^{(i)} & 0 & \bar{A}^{(i)T}\Phi & \bar{C}^{(i)T} \\ 0 & -\gamma^2 I & \bar{B}^{(i)T}\Phi & \bar{D}^{(i)T} \\ \Phi^T\bar{A}^{(i)} & \Phi^T\bar{B}^{(i)} & Q^{(i)} - (\Phi + \Phi^T) & 0 \\ \bar{C}^{(i)} & \bar{D}^{(i)} & 0 & -I \end{bmatrix} < 0. \tag{3.4.2}$$

The following result gives a solution to the H_∞ control problem.

Theorem 3.4.1 *The H_∞ control of the system (3.2.1)-(3.2.3) via dynamic output feedback can be solved if for some scalar $\varepsilon > 0$, there exists a solution ($\bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, Y, \bar{X}, \bar{Z}$) to the following LMIs:*

$$\Psi(\gamma^2 I, \bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z})|_{\bar{\Lambda}_1^{(i)}=\bar{\Lambda}_2^{(i)}=0} < 0$$

$$\Gamma(\bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, Y, \bar{X}, \varepsilon) > 0$$

for $i = 1, 2, \cdots, M$. In this situation, the parameters of a feasible H_∞ controller can be obtained by (3.3.30)-(3.3.32).

Remark 3.4.1 Observe that for a given ε, (3.3.28) and (3.3.29) are linear in ($S, \bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}$) and hence can be solved by employing the LMI Toolbox. Then the problem is how to find the optimal value of ε in order to optimize the H_2 or H_∞ norm. The numerical optimization algorithm **fminsearch** in the Optimization Toolbox of Matlab can be employed to give a locally convergent solution to the problem.

3.5 Illustrative Example

Consider a system which belongs to the 2-polytopic convex polyhedron in the form of (3.2.4), where

$$A^{(1)} = \begin{bmatrix} 0.1 & -0.2 \\ 0 & 1 \end{bmatrix}, \; A^{(2)} = \begin{bmatrix} 0.2 & -0.2 \\ 0 & 1 \end{bmatrix}, \; B_2^{(1)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$B_2^{(2)} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \; B_1^{(1)} = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}, \; B_1^{(2)} = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix},$$

$$C_2^{(1)} = C_2^{(2)} = [\,1 \quad 2\,], \; D_{21}^{(1)} = D_{21}^{(2)} = [\,0.2\,], \; C_1^{(1)} = [\,1 \quad 0\,],$$

$$C_1^{(2)} = [\,1 \quad 0.5\,], \; D_{11}^{(1)} = 0.1, \; D_{11}^{(2)} = 0.2, \; D_{12}^{(1)} = 1, \; D_{12}^{(2)} = 2.$$

We first study the H_2 performance. Letting $\varepsilon = 0.2$ and following Theorem 3.3.1, we have

$$\bar{X} = \begin{bmatrix} 0.3249 & 0.0914 \\ 0.0914 & 0.1004 \end{bmatrix}, \; Y = \begin{bmatrix} 0.3457 & 0.0204 \\ 0.0204 & 0.3425 \end{bmatrix}, \; \bar{Z} = \begin{bmatrix} 0.0300 & 0.0107 \\ 0.0063 & 0.0067 \end{bmatrix},$$

$$U = \begin{bmatrix} -0.1093 & 0.0894 \\ 0.3689 & -0.30180 \end{bmatrix}, \; W = \begin{bmatrix} 0.0276 \\ -0.0931 \end{bmatrix}, \; V = [\, -0.0591 \quad -0.1379 \,].$$

M_1 is chosen as I, then

$$N_1 = \bar{X}^{-T}(\bar{Z} - Y) = \begin{bmatrix} -1.2538 & 1.2261 \\ 1.0019 & -4.4620 \end{bmatrix}.$$

Therefore we obtain the controller of the form (3.2.5)-(3.2.6) with

$$\hat{A} = \begin{bmatrix} 0.5098 & 0.0327 \\ -1.4807 & -0.0951 \end{bmatrix}, \; \hat{B} = \begin{bmatrix} 0.4653 \\ -1.3515 \end{bmatrix}, \; \hat{C} = [\, 0.0349 \quad 0.0204 \,], \; \hat{D} = -0.0328,$$

which gives the H_2 norm bound of 1.8147.

In the following we will use the numerical optimization algorithm **fminsearch** in the Optimization Toolbox of Matlab to obtain a local optimal upper bound of the H_2 norm. Starting from the initial value $\varepsilon_0 = 0.2$, we arrive at the minimum value of 1.2435 with $\varepsilon = 0.6138$. And the resultant controller is given by

$$\hat{A} = \begin{bmatrix} 0.5390 & 0.0269 \\ -1.5951 & -0.0796 \end{bmatrix}, \; \hat{B} = \begin{bmatrix} 0.1414 \\ -0.4185 \end{bmatrix}, \; \hat{C} = [\, 0.1161 \quad 0.0745 \,], \; \hat{D} = -0.0327.$$

The actual H_2 performance under the resulted controller above is shown in Figure 3.1 when θ_1 varies from 0 to 1. For $\theta_1 = 0.8$, the evolution of z is shown in Figure 3.2 when a white noise signal is used as an input.

To reduce the conservatism in the above results, we will use the SLPMM method in Theorem 3.3.2. By setting $\epsilon = 1e - 4$, we obtain the optimal H_2 performance with

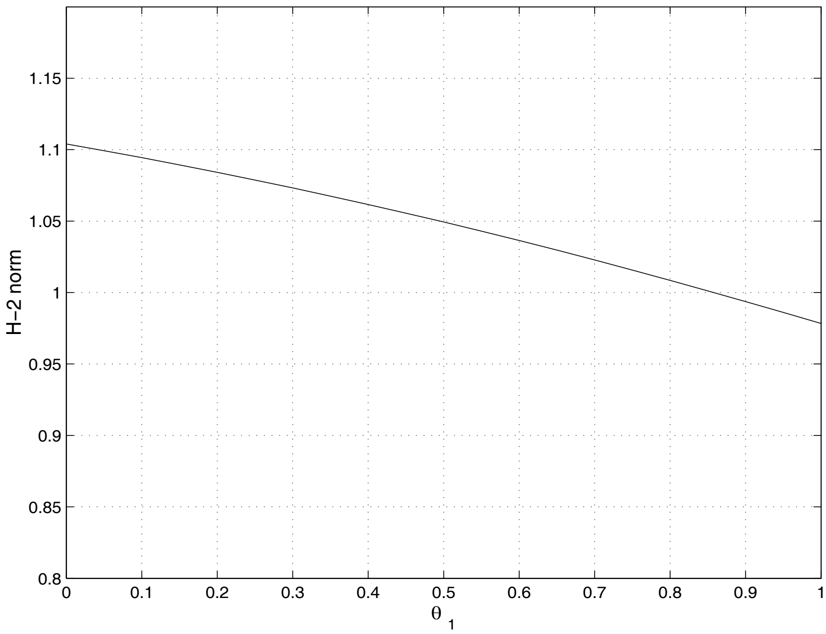


Figure 3.1 Actual H_2 norm versus θ_1

1.0796 and the following resulted controller

$$\hat{A} = \begin{bmatrix} 1.0148 & 0.1749 \\ -3.2915 & -0.8678 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0.1683 \\ -0.5032 \end{bmatrix}, \ \hat{C} = [\,-0.0661 \quad -0.0305\,], \ \hat{D} = -0.0327.$$

The actual H_2 performance under the above resulted controller is shown in Figure 3.3 when θ_1 varies from 0 to 1, from which we can see that the SLPMM method is less conservative than the previous two methods.

Letting $\varepsilon_0 = 0.4$ and applying Theorem 3.4.1 and the **fminsearch**, we can obtain an upper bound of the H_∞ norm of 3.1366 with $\varepsilon = 0.2170$. The following controller parameters can be obtained:

$$\hat{A} = \begin{bmatrix} 0.1802 & -0.1156 \\ -0.3657 & 0.2611 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0.0588 \\ -0.5884 \end{bmatrix}, \ \hat{C} = [\,-0.0008 \quad 0.0037\,], \ \hat{D} = -0.0590.$$

The frequency responses are shown in Figure 3.4 when θ_1 varies from 0 to 1, which verifies that the H_∞ norm of each case is below the bound.

Now we will discuss the H_∞ performance by using the SLPMM method. Setting

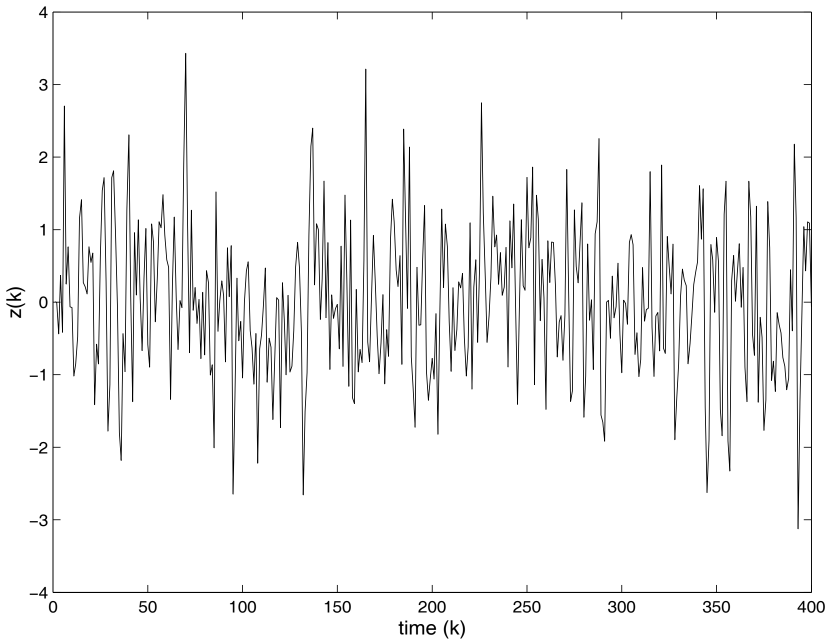


Figure 3.2 Evolution of $z(k)$

$\epsilon = 1e - 4$, we obtain the optimal H_∞ bound of 1.2307 which is much smaller than 3.1366. The resultant controller parameters are given by

$$\hat{A} = \begin{bmatrix} 0.5305 & -0.0893 \\ -2.7839 & -0.0838 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0.0199 \\ -0.4772 \end{bmatrix}, \ \hat{C} = [\,0.0388 \quad 0.0072\,], \ \hat{D} = -0.0355.$$

The frequency responses under the above controller are shown in Figure 3.5, which shows that the SLPMM method is much less conservative than the method of using Fminsearch.

3.6 Conclusion

In this chapter, we have developed a technique of solving the H_2 and H_∞ control problems for discrete-time systems with polytopic uncertainties via dynamic output feedback, which has not been solved in existing literature. A unified treatment of H_2 and H_∞ design is proposed. Based upon the technique of change of variables, the

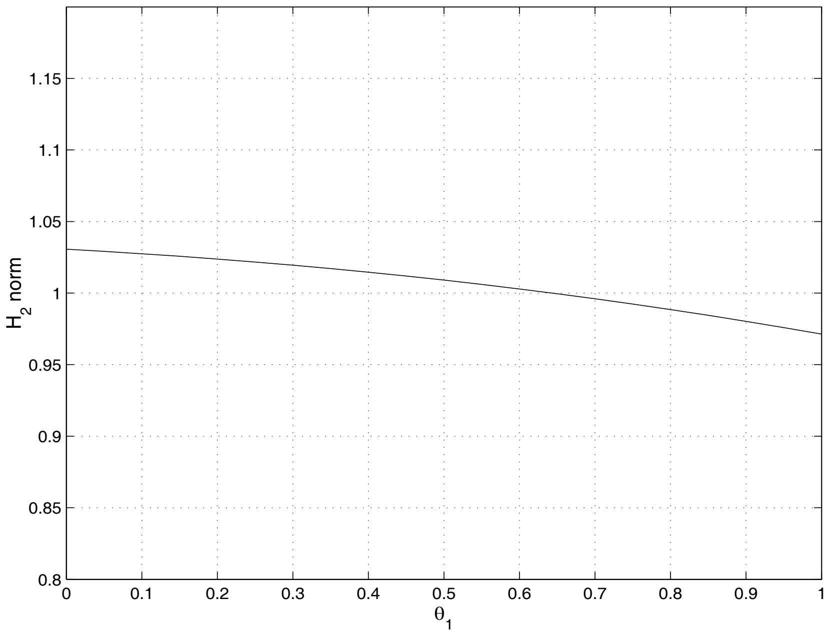


Figure 3.3 Actual H_2 norm versus θ_1, SLPMM method

H_2 and H_∞ controllers have been obtained in terms of LMIs which can be efficiently computed by existing software.

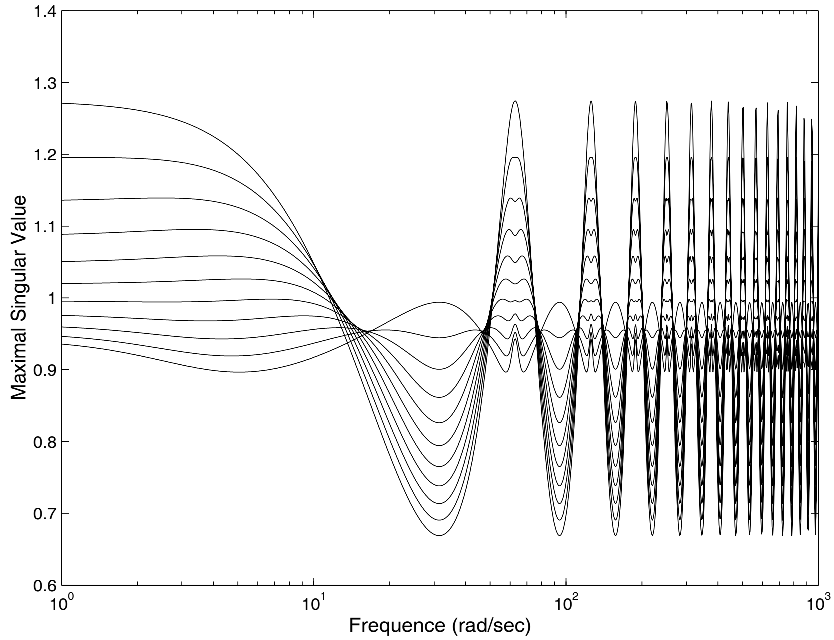


Figure 3.4 The frequency responses, under the resulted controller, of 11 systems with $\theta_1 \in [0\ 1]$.

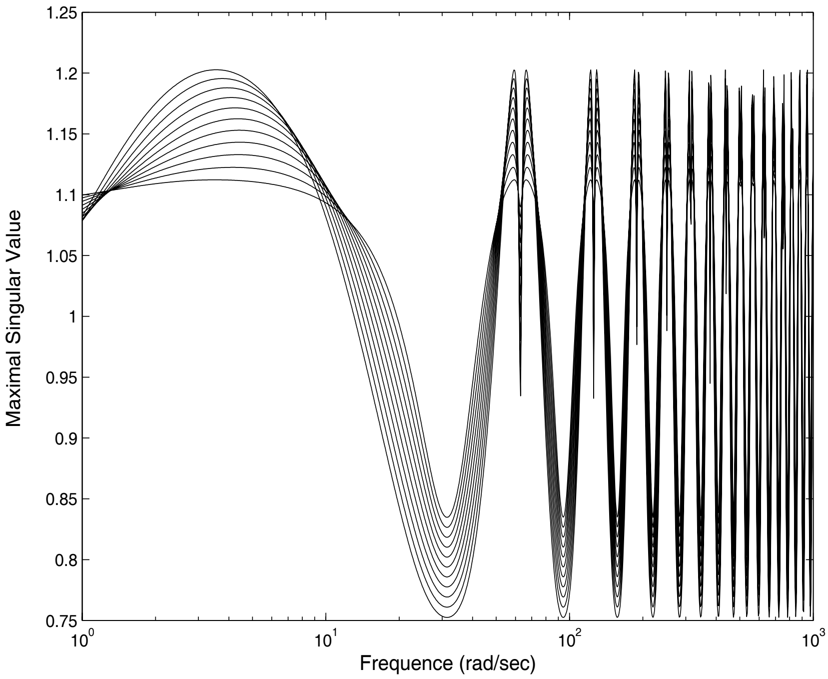


Figure 3.5 The frequency responses, under the resulted controller, of 11 systems with $\theta_1 \in [0\ 1]$.

Chapter 4

Improved Robust H_2 and H_∞ Filtering for Uncertain Discrete-time Systems

4.1 Introduction

The robust H_2 and H_∞ filtering problems have received a lot of interest in the past decades. There are basically two approaches to the problems, namely, the ARE approach and the LMI approach. The former was commonly adopted in dealing with the norm-bounded uncertainty in the early stage of development; see [XdSF91, PM96,XSdS94]. Recently, there has been much interest in the LMI approach mainly due to its numerical capability in handling more general type of uncertainty such as the polytopic uncertainty and solving multi-objective filtering problems; see [LF97, GdOB02,dOGB99,SXS01,TAN01]. In particular, a parameter dependent Lyapunov function based approach has been proposed for the robust H_2 filtering in [SXS01, GdOB02] for the case where a strictly proper filter is considered.

Since these results are all sufficient, attempts are being made towards improving the conservativeness of the design. In this chapter we explore the result of [PABB00] to deal with robust filtering problems for discrete-time systems with polytopic uncertainty. We show how this result can be extended to give a less conservative design than those of [SXS01,GdOB02,dOGB99]. Our solutions are given in terms of parameterized LMIs. As compared to existing results, two additional scaling parameters are introduced for the robust H_2 filtering and three additional scaling parameters and one matrix variable for the robust H_∞ filtering. These extra freedoms lead to a less conservative design than those of [SXS01, GdOB02]. In fact, the proposed approach will reduce to the results of [SXS01, GdOB02] when those free parameters are set to be zero. Hence, it is clear that a better filter can be designed by optimization over these parameters. An iterative approach is also proposed for further refinement of the robust H_2 filter. In particular, the iterative procedure indeed gives much improvement on filtering performance in some applications.

4.2 Problem formulation

Consider the following asymptotically stable system:

$$x(k+1) = Ax(k) + Bw(k), \qquad x(0) = 0 \tag{4.2.1}$$

$$y(k) = Cx(k) + Dw(k) \tag{4.2.2}$$

$$z(k) = L_1 x(k) + L_2 w(k) \tag{4.2.3}$$

where $x(k) \in \mathcal{R}^n$ is the system state vector, $y(k) \in \mathcal{R}^r$ is the measurement, $z(k) \in \mathcal{R}^q$ is the signal to be estimated and $w(k) \in \mathcal{R}^p$ is the noise input.

Note that for the case when the process noise and input noise are different (usually so in practice), say $w_1(k)$ and $w_2(k)$, we can simply put $B = [B_1 \;\; 0]$, $D = [0 \;\; D_1]$, $L_2 = [L_{21} \;\; 0]$ and let $w(k) = [w_1(k)^T \;\; w_2(k)^T]^T$ in the system model (4.2.1)-(4.2.3).

The matrices $A,\ B,\ C,\ D,\ L_1$ and L_2 are appropriately dimensioned with partially unknown parameters. They belong to the following uncertainty polytope:

$$\begin{aligned}\Omega &= \{(A, B, C, D, L_1, L_2) \mid (A, B, C, D, L_1, L_2) = \\ &\left.\sum_{i=1}^{M} \theta_i (A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}, L_1^{(i)}, L_2^{(i)}),\ \theta_i \geq 0,\ \sum_{i=1}^{M} \theta_i = 1 \right\}\end{aligned} \tag{4.2.4}$$

We consider a filter of the form for the system (4.2.1)-(4.2.3):

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}y(k), \quad \hat{x}(0) = 0; \tag{4.2.5}$$

$$\hat{z}(k) = \hat{C}\hat{x}(k) + \hat{D}y(k) \tag{4.2.6}$$

where the matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ are to be determined.

Robust H_2 filtering problem: Assume that the noise input $w(k)$ is a Gaussian white noise with zero-mean and unit covariance. Design a filter of the form (4.2.5)-(4.2.6) such that for all uncertainties belonging to the polytope (4.2.4) the filtering error variance, $\mathcal{E}\{[z(k) - \hat{z}(k)]^T [z(k) - \hat{z}(k)]\}$, has a minimum possible upper bound.

Robust H_∞ filtering problem: Assume that $w \in \ell_2[0, \infty)$. Given a prescribed scalar γ, design a filter of the form (4.2.5)-(4.2.6) such that for all non-zero $w \in \ell_2[0, \infty)$,

$$\|z - \hat{z}\|_2 \leq \gamma \|w\|_2$$

over the entire polytope Ω.

4.3 Robust H_2 Filter

This section first presents a less conservative analysis result for evaluating the upper bound of the H_2 norm of uncertain discrete-time systems. Additional free parameters (slack variables) are introduced in the result which help reduce the conservatism of the evaluation. We then apply the result to derive a less conservative design for the robust H_2 filter.

First, denote $\xi(k) = [x^T(k) \quad \hat{x}^T(k)]^T$. It follows from (4.2.1)-(4.2.3) and (4.2.5)-(4.2.6) that

$$\begin{aligned} \xi(k+1) &= \bar{A}\xi(k) + \bar{B}w(k) \quad (4.3.1)\\ z(k) - \hat{z}(k) &= \bar{C}\xi(k) + \bar{D}w(k) \quad (4.3.2)\end{aligned}$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & 0 \\ \hat{B}C & \hat{A} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ \hat{B}D \end{bmatrix}, \quad (4.3.3)\\ \bar{C} &= \begin{bmatrix} L_1 - \hat{D}C & -\hat{C} \end{bmatrix}, \quad \bar{D} = L_2 - \hat{D}D. \quad (4.3.4)\end{aligned}$$

Recall that when the matrices $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are known, the H_2 norm of the system (4.3.1)-(4.3.2) can be computed by the following minimization [GdOB02, SXS01]:

$$\min_P trace(\bar{C}P\bar{C}^T + \bar{D}\bar{D}^T) \quad (4.3.5)$$

subject to

$$\bar{A}P\bar{A}^T - P + \bar{B}\bar{B}^T < 0. \quad (4.3.6)$$

Note that (4.3.6) is equivalent to

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} diag\{P, I\} \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} - P < 0 \quad (4.3.7)$$

or

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix}^T Q \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} - diag\{Q, I\} < 0 \quad (4.3.8)$$

where $Q = P^{-1}$.

The following lemma can be established; see also [PABB00].

Lemma 4.3.1 *There exists a matrix $Q = Q^T > 0$ to (4.3.8) if and only if there exists a solution (F, G, Q) with $Q = Q^T$ such that*

$$\begin{bmatrix} -diag\{Q, I\} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} F + F^T[\bar{A} \quad \bar{B}] & -F^T + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} G \\ -F + G^T[\bar{A} \quad \bar{B}] & Q - (G + G^T) \end{bmatrix} < 0. \tag{4.3.9}$$

Proof The proof is rather straightforward. First, if (4.3.8) holds for some $Q > 0$, by setting $F = 0$ and $G^T = G = Q$ and applying the Schur complement, (4.3.9) is satisfied. On the other hand, if (4.3.9) holds for some (F, Q, G), multiplying (4.3.9) from the left and from the right by Γ^T and Γ, respectively, where

$$\Gamma = \begin{bmatrix} I \\ [\bar{A} \quad \bar{B}] \end{bmatrix},$$

(4.3.8) follows.

Remark 4.3.1 When the matrices $(\bar{A}, \bar{B})$ are known, the above result implies the equivalence between (4.3.8) and (4.3.9). However, if the matrices $(\bar{A}, \bar{B})$ are from an uncertain polytope, (4.3.9) would render a less conservative evaluation of the upper-bound of the H_2 norm of the system (4.3.1)-(4.3.2) due to the freedom given by the slack variables F and G and the fact that Q is allowed to be vertex-dependent in (4.3.9). We note that when setting $F = 0$, Lemma 4.3.1 reduces to that in [SXS01, GdOB02]. This additional matrix variable will enable us to derive a less conservative design than those of [SXS01, GdOB02].

Lemma 4.3.2 *Given a filter $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, an upper bound of the H_2 norm of the error system (4.3.1)-(4.3.2) can be evaluated by the following optimization:*

$$\min_{(F, G, Q^{(i)}, i=1,2,\cdots,N)} trace(S)$$

subject to

$$\begin{bmatrix} -diag\{Q^{(i)}, I\} + \begin{bmatrix} \bar{A}^{(i)T} \\ \bar{B}^{(i)T} \end{bmatrix} F + F^T[\bar{A}^{(i)} \quad \bar{B}^{(i)}] & -F^T + \begin{bmatrix} \bar{A}^{(i)T} \\ \bar{B}^{(i)T} \end{bmatrix} G \\ -F + G^T[\bar{A}^{(i)} \quad \bar{B}^{(i)}] & Q^{(i)} - (G + G^T) \end{bmatrix} < 0 \tag{4.3.10}$$

and

$$\begin{bmatrix} S & \bar{C}^{(i)} & \bar{D}^{(i)} \\ \bar{C}^{(i)T} & Q^{(i)} & 0 \\ \bar{D}^{(i)T} & 0 & I \end{bmatrix} > 0 \tag{4.3.11}$$

for $i = 1, 2, \cdots, M$, where $\bar{A}^{(i)}$, $\bar{B}^{(i)}$, $\bar{C}^{(i)}$ and $\bar{D}^{(i)}$ are the matrices in (4.3.3)-(4.3.4) at the i-th vertex of the polytope Ω.

Proof First, observe that a convex combination of (4.3.10) at all the vertices of Ω implies (4.3.9) for $Q = \sum \theta_i Q^{(i)}$ by considering the fact that $[\bar{A} \ \ \bar{B}] = \sum \theta_i [\bar{A}^{(i)} \ \ \bar{B}^{(i)}]$. Hence, by Lemma 4.3.1, (4.3.8) holds for $Q = \sum \theta_i Q^{(i)}$. Similarly, (4.3.11) implies by the Schur complement that $\bar{C}(\sum \theta_i Q^{(i)})^{-1} \bar{C}^T + \bar{D}\bar{D}^T < S$. Hence, the optimization in Lemma 4.3.2 is equivalent to that in (4.3.5) subject to (4.3.8) with $P = (\sum \theta_i Q^{(i)})^{-1}$.

The proof of Lemma 4.3.2 clearly demonstrates the use of a parameter-dependent Lyapunov function.

While the above is useful for evaluating the H_2 norm bound for the error system (4.3.1)-(4.3.2) when a filter (4.2.5)-(4.2.6) is given, it may not be directly applicable to the robust H_2 filter design problem due to the presence of the products of F with $\bar{A}^{(i)}$ and G with $\bar{A}^{(i)}$. To enable the sub-optimal robust H_2 filter design, we specialize the matrix F as follows:

$$F = \begin{bmatrix} \Lambda G & 0_{2n \times m} \end{bmatrix} \tag{4.3.12}$$

where $\Lambda = diag\{\lambda_1 I_n, \lambda_2 I_n\}$ with λ_1 and λ_2 being real scalars.

Using the above F, (4.3.10) can be rewritten as

$$\begin{bmatrix} -Q^{(i)} + \bar{A}^{(i)T}\Lambda G + G^T\Lambda\bar{A}^{(i)} & G^T\Lambda\bar{B}^{(i)} & -G^T\Lambda + \bar{A}^{(i)T}G \\ \bar{B}^{(i)T}\Lambda G & -I & \bar{B}^{(i)T}G \\ -\Lambda G + G^T\bar{A}^{(i)} & G^T\bar{B}^{(i)} & Q^{(i)} - (G + G^T) \end{bmatrix} < 0. \qquad (4.3.13)$$

The following result gives a solution to the robust H_2 filtering problem.

Theorem 4.3.1 *Consider the system (4.2.1)-(4.2.3) over the polytope (4.2.4). A filter of the form (4.2.5)-(4.2.6) that gives a suboptimal guaranteed filtering error covariance bound can be derived from the following optimization:*

$$\min_{(R,W,S_A,S_B,S_C,S_D,T,\hat{Q}_{11}^{(i)},\hat{Q}_{12}^{(i)},\hat{Q}_{22}^{(i)},i=1,2,\cdots N,\lambda_1,\lambda_2)} \quad trace(S)$$

subject to

$$\begin{bmatrix} \lambda_1(A^{(i)T}R + R^TA^{(i)}) - \hat{Q}_{11}^{(i)} & * & * \\ \lambda_1 W^TA^{(i)} + \lambda_2(S_BC^{(i)} + S_A) - \hat{Q}_{12}^{(i)T} & -\lambda_2(S_A + S_A^T) - \hat{Q}_{22}^{(i)} & * \\ \lambda_1 B^{(i)T}R & \lambda_1 B^{(i)T}W + \lambda_2 D^{(i)T}S_B^T & -I \\ R^TA^{(i)} - \lambda_1 R & -\lambda_1 W - \lambda_2 T^T & R^TB^{(i)} \\ W^TA^{(i)} + S_BC^{(i)} + S_A & -S_A + \lambda_2 T^T & W^TB^{(i)} + S_BD^{(i)} \end{bmatrix.$$

$$\begin{bmatrix} * & * \\ * & * \\ * & * \\ \hat{Q}_{11}^{(i)} - (R + R^T) & * \\ \hat{Q}_{12}^{(i)T} - (W^T + T) & \hat{Q}_{22}^{(i)} + (T + T^T) \end{bmatrix} < 0 \qquad (4.3.14)$$

and

$$\begin{bmatrix} S & * & * & * \\ L_1^{(i)T} - C^{(i)T}S_D^T - S_C^T & \hat{Q}_{11}^{(i)} & * & * \\ S_C^T & \hat{Q}_{12}^{(i)T} & \hat{Q}_{22}^{(i)} & * \\ L_2^{(i)T} - D^{(i)T}S_D^T & 0 & 0 & I \end{bmatrix} > 0 \qquad (4.3.15)$$

for $i = 1, 2, \cdots, M$. The suboptimal filter is given by

$$\hat{A} = T^{-1}S_A, \quad \hat{B} = T^{-1}S_B, \quad \hat{C} = S_C, \quad \hat{D} = S_D. \qquad (4.3.16)$$

Proof First, observe from (4.3.13) that G is invertible since $G + G^T > Q^{(i)} > 0$. Denote

$$G = \begin{bmatrix} X & \bar{M} \\ M_1 & U \end{bmatrix}, \; G^{-1} = \begin{bmatrix} Y & \bar{N} \\ N_1 & V \end{bmatrix}, \; J = \begin{bmatrix} Y & I_n \\ N_1 & 0 \end{bmatrix}, \; J_1 = diag\{J, I, J\}.$$

Multiply from the left and the right of (4.3.13) by J_1^T and J_1 respectively and apply (4.3.3). We obtain

$$\begin{bmatrix} \lambda_1(A^{(i)}Y + Y^TA^{(i)T}) - \bar{Q}_{11}^{(i)} & * & * & * & * \\ \Delta_1^{(i)} & \Delta_2^{(i)} & * & * & * \\ \lambda_1 B^{(i)T} & \lambda_1 B^{(i)T}X + \lambda_2 D^{(i)T}\hat{B}^TM_1 & -I & * & * \\ A^{(i)}Y - \lambda_1 Y^T & A^{(i)} - (\lambda_1 Y^TX + \lambda_2 N_1^TM_1) & B^{(i)} & \bar{Q}_{11}^{(i)} - (Y + Y^T) & * \\ -\lambda_1 I + X^TA^{(i)}Y + M_1^T(\hat{B}C^{(i)}Y + \hat{A}N_1) & -\lambda_1 X + X^TA^{(i)} + M_1^T\hat{B}C^{(i)} & X^TB^{(i)} + M_1^T\hat{B}D^{(i)} & \bar{Q}_{12}^{(i)T} - (I + X^TY + M_1^TN) & \bar{Q}_{22}^{(i)} - (X + X^T) \end{bmatrix} < 0 \qquad (4.3.17)$$

where $\Delta_1^{(i)} = \lambda_1(A^{(i)T}+X^TA^{(i)}Y)+\lambda_2 M_1^T(\hat{B}C^{(i)}Y+\hat{A}N_1)-\bar{Q}_{12}^{(i)T}$, $\Delta_2^{(i)} = \lambda_1(A^{(i)T}X+X^TA^{(i)}) + \lambda_2(C^{(i)T}\hat{B}^TM_1 + M_1^T\hat{B}C^{(i)}) - \bar{Q}_{22}^{(i)}$ and $\bar{Q}^{(i)} = [\bar{Q}_{lj}^{(i)}] = J^TQ^{(i)}J$.

Next, it is clear from (4.3.17) that Y is invertible. So, let $R = Y^{-1}$ and

$$\hat{Q}^{(i)} = \begin{bmatrix} \hat{Q}_{11}^{(i)} & \hat{Q}_{12}^{(i)} \\ \hat{Q}_{12}^{(i)T} & \hat{Q}_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} R & -R \\ 0 & I \end{bmatrix}^T \bar{Q}^{(i)} \begin{bmatrix} R & -R \\ 0 & I \end{bmatrix}.$$

Also, define

$$S_A = M_1^T\hat{A}N_1R, \quad S_B = M_1^T\hat{B}, \quad S_D = \hat{D}, \quad T = M_1^TN_1R, \quad W = X - R. \quad (4.3.18)$$

Then, (4.3.14) follows by multiplying (4.3.17) from the left and from the right by J_2^T and J_2, respectively, where

$$J_2 = diag\left\{\begin{bmatrix} R & -R \\ 0 & I \end{bmatrix}, I, \begin{bmatrix} R & -R \\ 0 & I \end{bmatrix}\right\}.$$

Similarly, (4.3.15) can be obtained by multiplying (4.3.11) from the left and from the right by $diag\{I, J^T, I\}$ and $diag\{I, J, I\}$, then further by J_3^T and J_3, respectively, where

$$J_3 = diag\left\{I, \begin{bmatrix} R & -R \\ 0 & I \end{bmatrix}, I\right\}.$$

Note that it is clear from (4.3.14) that T is invertible, so are M_1, N_1 and R in view of (4.3.18). Hence, it follows from (4.3.18) that the filter is of the transfer function:

$$F(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} = S_CR^{-1}N_1^{-1}(zI - M_1^{-T}S_AR^{-1}N_1^{-1})^{-1}M_1^{-T}S_B + S_D$$

$$= S_C(zT - S_A)^{-1}S_B + S_D = S_C(zI - T^{-1}S_A)^{-1}T^{-1}S_B + S_D$$

which gives the filter parameters of (4.3.16).

Remark 4.3.2 It should be mentioned that when $\lambda_1 = \lambda_2 = 0$ and $S_D = 0$, Theorem 4.3.1 recovers the existing results in [SXS01, GdOB02] where the signal to be estimated does not explicitly contain the input noise w and a strictly proper filter is adopted. It is thus expected that the result in Theorem 4.3.1 should be less conservative due to the extra degrees of freedom in optimization. The numerical example in Section 4.5 will verify this fact.

Remark 4.3.3 Observe that for given λ_1 and λ_2, (4.3.14) and (4.3.15) are linear in R, W, S_A, S_B, S_C, S_D, T, $\hat{Q}_{11}^{(i)}$, $\hat{Q}_{12}^{(i)}$ and $\hat{Q}_{22}^{(i)}$, and hence can be solved by employing the LMI Tool [GNLC95]. The problem is then how to find the optimal values of λ_1 and λ_2 in order to minimize the filtering error variance bound. One way to address the tuning issue is to first solve the feasibility problem of the LMIs (4.3.14)-(4.3.15) with $i = 1, 2, \cdots, M$ using Matlab's LMI toolbox [GNLC95] and obtain a set of initial scaling parameters. Then, applying a numerical optimization algorithm, such as the program **fminsearch** in the optimization toolbox of Matlab [CBG99], a locally convergent solution to the problem is obtained. This optimization procedure is efficient for the optimization in Theorem 4.3.1 as it only involves the search of two parameters λ_1 and λ_2, which is also known from our experience in numerical examples.

Note that Theorem 4.3.1 has been derived with a specialized matrix F of the form (4.3.12) in order to linearize the matrix inequality. This, however, is restrictive. In the following we will propose an iterative LMI algorithm which can be applied to refine the filter designed using Theorem 4.3.1.

To this end, we denote

$$\overline{AB}^{(i)} = \begin{bmatrix} A^{(i)} & 0 & B^{(i)} \\ 0 & 0 & 0 \end{bmatrix}, \quad \widehat{AB} = [\,\hat{A} \quad \hat{B}\,], \quad \overline{CD} = \begin{bmatrix} 0 & I & 0 \\ C^{(i)} & 0 & D^{(i)} \end{bmatrix}$$

and $\Xi^{(i)} = \overline{AB}^{(i)T} F + \overline{CD}^{(i)T} \widehat{AB}^T [0 \quad I] F$. Then, (4.3.10) can be rewritten as

$$\begin{bmatrix} -diag\{Q^{(i)}, I\} + \overline{AB}^{(i)T} F + F^T \overline{AB}^{(i)} + \Xi^T + \Xi & * \\ -F + G^T \overline{AB} + G^T \begin{bmatrix} 0 \\ I \end{bmatrix} \widehat{AB}\overline{CD}^{(i)} & Q^{(i)} - (G + G^T) \end{bmatrix} < 0. \tag{4.3.19}$$

The following iterative procedure can be applied:

- Step 1: Given the filter parameters $(\hat{A}, \hat{B})$, F, G, $Q^{(i)}$, $\hat{C}$ and $\hat{D}$ may be found by minimizing $trace(S)$ subject to (4.3.10) and (4.3.11). The initial $(\hat{A}, \hat{B})$ can be the suboptimal filter designed by Theorem 4.3.1.
- Step 2: With the F, G and $Q^{(i)}$ obtained in Step 1, an improved filter can be obtained by minimizing $trace(S)$ subject to (4.3.19) and (4.3.11).
- Repeat the above steps until $trace(S_{k-1} - S_k) < \mu$, where μ is a prescribed tolerance and S_k is the matrix S of (4.3.11) at the k-th iteration.

It should be emphasized that the above iteration always converges.

4.4 Robust H_∞ Filter

In this section, we shall extend the technique of the previous section to solve the robust H_∞ filtering problem.

Recall that when the system (4.3.1) and (4.3.2) is known, it is stable with its H_∞ norm less than γ if and only if there exists a matrix $P = P^T > 0$ such that [ZKSN92, Xie96]

$$\tilde{A}^T diag\{P, I\} \tilde{A} - diag\{P, \gamma^2 I\} < 0 \tag{4.4.1}$$

where

$$\tilde{A} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}.$$

As in Chapter 3, we also assume $p = q$, i.e., the input and the signal to be estimated have the same dimension. If this is not the case, the modification method in Chapter 3 can be applied again.

We have the following result. Its proof is similar to Lemma 4.3.1 and is thus omitted.

Lemma 4.4.1 *When the system (4.3.1) and (4.3.2) is known, it is stable with its H_∞ norm less than γ if and only if there exist matrices (P, F, G) with $P = P^T$ such that*

$$\begin{bmatrix} -diag\{P, \gamma^2 I\} + \tilde{A}^T F + F^T \tilde{A} & -F^T + \tilde{A}^T G \\ -F + G^T \tilde{A} & diag\{P, I\} - (G + G^T) \end{bmatrix} < 0. \quad (4.4.2)$$

Note that any solution P of (4.4.2) must be positive definite since (4.4.2) implies (4.4.1) which clearly indicates the positive definiteness of P.

As commented in Section 4.3, the above result is useful for H_∞ analysis when the filter is given. In order to facilitate the robust H_∞ filter design, we need to consider a special case of the above lemma. To this end, we specify the matrices F and G as follows:

$$F = \begin{bmatrix} \Lambda\Phi & 0 \\ 0 & \varepsilon\Psi \end{bmatrix}, \quad G = \begin{bmatrix} \Phi & 0 \\ 0 & \Psi \end{bmatrix} \quad (4.4.3)$$

where $\Phi \in \mathcal{R}^{n\times n}$, $\Psi \in \mathcal{R}^{q\times q}$, $\Lambda = diag\{\lambda_1 I_n, \lambda_2 I_n\}$ and λ_1, λ_2 and ε are any real numbers. Substituting (4.4.3) into (4.4.2) and applying (4.3.3) and (4.3.4) lead to

$$\begin{bmatrix} -P+\bar{A}^T\Lambda\Phi+\Phi^T\Lambda\bar{A} & \varepsilon\bar{C}^T\Psi+\Phi^T\Lambda\bar{B} & -\Phi^T\Lambda+\bar{A}^T\Phi & \bar{C}^T\Psi \\ \varepsilon\Psi^T\bar{C}+\bar{B}^T\Lambda\Phi & -I+\varepsilon\left(\bar{D}^T\Psi+\Psi^T\bar{D}\right) & \bar{B}^T\Phi & -\varepsilon\Psi^T+\bar{D}^T\Psi \\ -\Lambda\Psi+\Phi^T\bar{A} & \Phi^T\bar{B} & P-(\Phi+\Phi^T) & 0 \\ \Psi^T\bar{C} & -\varepsilon\Psi+\Psi^T\bar{D} & 0 & I-(\Psi+\Psi^T) \end{bmatrix} < 0. \tag{4.4.4}$$

Remark 4.4.1 When setting $\lambda_1=\lambda_2=\varepsilon=0$ (i.e., setting $F=0$) and $\Psi=I$ and by some row-column exchanges, the above inequality reduces to

$$\begin{bmatrix} P-\Phi-\Phi^T & \Phi^T\bar{A} & \Phi^T\bar{B} & 0 \\ \bar{A}^T\Phi & -P & 0 & \bar{C}^T \\ \bar{B}^T\Phi & 0 & -\gamma^2 I & \bar{D}^T \\ 0 & \bar{C} & \bar{D} & -I \end{bmatrix} < 0$$

which is the result of [dOGB99]. Therefore, our result in Lemma 4.4.1 should lead to a less conservative result due to the additional freedoms given by the scalars λ_1, λ_2, ε and the matrix variable Ψ. We note that (4.4.4) implies that Ψ is invertible.

Theorem 4.4.1 *Consider the system (4.2.1)-(4.2.3) over the polytope (4.2.4). A filter of the form (4.2.5)-(4.2.6) that solves the robust H_∞ filtering problem exists if for some λ_1, λ_2 and ε, there exists a solution $(\hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, i=1,2,\cdots,N; R, W, S_A, S_B, S_C, S_D, T, \Psi)$ to the following LMIs:*

$$\left[\begin{matrix} \lambda_1(A^{(i)T}R+R^TA^{(i)})-\hat{P}_{11}^{(i)} & * \\ \lambda_1 W^TA^{(i)}+\lambda_2(S_BC^{(i)}+S_A)-\hat{P}_{12}^{(i)T} & -\lambda_2(S_A+S_A^T)-\hat{P}_{22}^{(i)} \\ \varepsilon(\Psi^TL_1^{(i)}-S_DC^{(i)}-S_C)+\lambda_1B^{(i)T}R & \varepsilon S_C+\lambda_1B^{(i)T}W+\lambda_2D^{(i)T}S_B^T \\ -\lambda_1R+R^TA^{(i)} & -\lambda_1W-\lambda_2T^T \\ W^TA^{(i)}+S_BC^{(i)}+S_A & \lambda_2T^T-S_A \\ \Psi^TL_1^{(i)}-S_DC^{(i)}-S_C & S_C \end{matrix}\right.$$

$$\left.\begin{matrix} * & * & * & * \\ * & * & * & * \\ \Gamma^{(i)} & * & * & * \\ R^T B^{(i)} & \hat{P}_{11}^{(i)} - (R + R^T) & * & * \\ W^T B^{(i)} + S_B D^{(i)} & \hat{P}_{12}^{(i)T} - (W^T + T) & \hat{P}_{22}^{(i)} + (T + T^T) & * \\ -\varepsilon\Psi + \Psi^T L_2^{(i)} - S_D D^{(i)} & 0 & 0 & I - (\Psi + \Psi^T) \end{matrix}\right] < 0 \tag{4.4.5}$$

for $i = 1, 2, \cdots, M$, *where* $\Gamma^{(i)} = -\gamma^2 I + \varepsilon(\Psi^T L_2^{(i)} + L_2^{(i)T}\Psi - S_D D^{(i)} - D^{(i)T} S_D^T)$. *In this situation, a suitable* H_∞ *filter is given by*

$$\hat{A} = T^{-1}S_A, \quad \hat{B} = T^{-1}S_B, \quad \hat{C} = \Psi^{-T}S_C, \quad \hat{D} = \Psi^{-T}S_D. \tag{4.4.6}$$

Proof The proof follows similarly to the proof of Theorem 4.3.1.

First, observe from (4.4.4) that Φ is invertible since $\Phi + \Phi^T > P > 0$. Denote

$$\Phi = \begin{bmatrix} X & \bar{M} \\ M_1 & U \end{bmatrix}, \quad \Phi^{-1} = \begin{bmatrix} Y & \bar{N} \\ N_1 & V \end{bmatrix}$$

and

$$J = \begin{bmatrix} Y & I_n \\ N_1 & 0 \end{bmatrix}, \quad J_4 = diag\{J, I_n, J, I_n\}.$$

Multiplying from the left and the right of (4.4.2) by J_4^T and J_4 respectively and applying (4.3.3)-(4.3.4), it follows that

$$\left[\begin{matrix} -\bar{P}_{11}^{(i)} + \lambda_1(A^{(i)}Y + Y^T A^{(i)T}) & * \\ \Theta_1^{(i)} & \Theta_2^{(i)} \\ \varepsilon\Psi^T(L_1^{(i)}Y - \hat{D}C^{(i)}Y - \hat{C}N_1) + \lambda_1 B^{(i)T} & \varepsilon\Psi^T(L_1^{(i)} - \hat{D}C^{(i)}) + \lambda_1 B^{(i)T}X + \lambda_2 D^{(i)T}\hat{B}^T M_1 \\ -\lambda_1 Y^T + A^{(i)}Y & -(\lambda_1 Y^T X + \lambda_2 N_1^T M_1) + A^{(i)} \\ -\lambda_1 I + X^T A^{(i)}Y + M_1^T \hat{B}C^{(i)}Y + M_1^T \hat{A}N_1 & -\lambda_1 X + X^T A^{(i)} + M_1^T \hat{B}C^{(i)} \\ \Psi^T(L_1^{(i)}Y - \hat{D}C^{(i)}Y - \hat{C}N_1) & \Psi^T(L_1^{(i)} - \hat{D}C^{(i)}) \end{matrix}\right.$$

$$
\begin{bmatrix}
\cdots & * & * & * & * \\
\cdots & * & * & * & * \\
\cdots & -\gamma^2 I + \varepsilon(\bar{D}^T\Psi + \Psi^T\bar{D}) & * & * & * \\
\cdots & B^{(i)} & \bar{P}_{11}^{(i)} - (Y + Y^T) & * & * \\
\cdots & X^T B^{(i)} + M_1^T \hat{B} D^{(i)} & \bar{P}_{12}^{(i)T} - (I + X^T Y + M_1^T N_1) & \bar{P}_{22}^{(i)} - (X + X^T) & * \\
\cdots & -\varepsilon\Psi + \Psi^T \bar{D} & 0 & 0 & I - (\Psi + \Psi^T)
\end{bmatrix} < 0
\quad (4.4.7)
$$

where $\Theta_1^{(i)} = -\bar{P}_{12}^{(i)T} + \lambda_1(A^{(i)T} + X^T A^{(i)} Y) + \lambda_2(M_1^T \hat{B} C^{(i)} Y + M_1^T \hat{A} N_1)$ and $\Theta_2^{(i)} = -\bar{P}_{22}^{(i)} + \lambda_1(X^T A^{(i)} + A^{(i)T} X) + \lambda_2(M_1^T \hat{B} C^{(i)} + C^{(i)T} \hat{B}^T M_1)$. Again, note that Y is invertible and let $R = Y^{-1}$. Further, multiply the matrix inequality (4.4.7) from the left and from the right by J_5^T and J_5, respectively, where

$$
J_5 = diag\left\{ \begin{bmatrix} R & -R \\ 0 & I \end{bmatrix}, I, \begin{bmatrix} R & -R \\ 0 & I \end{bmatrix}, I \right\}
$$

and denote

$$
S_A = M_1^T \hat{A} N_1 R,\ \ S_B = M_1^T \hat{B},\ \ S_C = \Psi^T \hat{C} N_1 R,\ \ S_D = \Psi^T \hat{D},\ \ W = X - R,\ \ T = M_1^T N_1 R.
$$

It can be verified that (4.4.5) follows immediately. Further, the transfer function of the filter is given by

$$
\begin{aligned}
F(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} &= \Psi^{-T} S_C R^{-1} N_1^{-1} (zI - M_1^{-T} S_A R^{-1} N_1^{-1})^{-1} M_1^{-T} S_B + \Psi^{-T} S_D \\
&= \Psi^{-T} S_C (zI - T^{-1} S_A)^{-1} T^{-1} S_B + \Psi^{-T} S_D.
\end{aligned}
$$

Hence, the filter parameters of (4.4.6) follows.

Remark 4.4.2 Observe that for given λ_1, λ_2 and ε, (4.4.6) is linear in $\hat{P}_{11}^{(i)}$, $\hat{P}_{12}^{(i)}$, $\hat{P}_{22}^{(i)}$, R, W, S_A, S_B, S_C, S_D, T, Ψ and can be solved by convex optimization. As for the problem of searching for optimal scaling parameters λ_1, λ_2 and ε, a similar procedure as mentioned in Remark 4.3.3 can be applied.

4.5 Illustrative Examples

4.5.1 The example in [XSdS94]

Consider the example in [XSdS94]:

$$x(k+1) = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\delta \end{bmatrix} x(k) + \begin{bmatrix} -6 \\ 1 \end{bmatrix} w(k) \quad (4.5.1)$$

$$y(k) = \begin{bmatrix} -100 & 10 \end{bmatrix} x(k) + v(k) \quad (4.5.2)$$

$$z(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \quad (4.5.3)$$

where $w(k)$ and $v(k)$ are uncorrelated zero-mean white noise signals with unit variances, respectively. δ is the uncertain parameter satisfying $|\delta| \leq \delta_0$, where δ_0 is known to be a positive real number. Obviously, the uncertainty in this system can be represented by a two-vertex polytope. We consider three cases $\delta_0 = 0.3, 0.4, 0.45$ respectively. The suboptimal upper bound of the filtering error variances are shown in Table 4.1.

	$\delta_0 = 0.3$	$\delta_0 = 0.4$	$\delta_0 = 0.45$
First method in [SXS01]	52.17	63.54	86.05
Iterative refinement in [SXS01]	51.40	58.78	72.97
Proposed Theorem 4.3.1	51.59	58.95	65.39
Proposed iterative method	51.43	57.57	61.31

Table 4.1 Performance comparison among different methods for example 4.5.1.

From the results shown in Table 4.1, it can be seen that the guaranteed filtering error bounds based upon the improved method of this chapter are smaller than those based upon the first method in [SXS01] for all the three cases. And the bounds obtained by the iterative method of this chapter are also smaller than those by the iterative method in [SXS01]. Our methods give better upper bounds for the guaranteed error covariance than the methods in [SXS01] especially when the bound of the uncertainty becomes larger. For the case of $\delta_0 = 0.45$, the minimum bound of 65.39 is obtained by using Theorem 4.3.1 for $\lambda_1 = -0.8$ and $\lambda_2 = 0.2$ and the filter parameters are given by

$$\hat{A} = \begin{bmatrix} -0.2506 & 0.0062 \\ -1.5257 & 0.8484 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} -0.0108 \\ -0.0039 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 0.6818 & -0.0172 \end{bmatrix}$$

where for the purpose of making a fair comparison with [SXS01], a strictly proper filter (one step shead predictor) has been adopted.

Starting from the above filter parameters, we can employ the iterative method and get the minimum bound of 61.31. The matrices F and G are obtained as

$$F = \begin{bmatrix} 0.1164 & 0.6280 & -0.2507 & 0.1335 & 7.5690 & -0.0009 \\ -0.0090 & -0.0188 & 0.0132 & -0.0014 & 0.0000 & -0.0000 \\ -7.0212 & 0.7743 & -0.1686 & 1.0091 & 0.3067 & 0.0625 \\ -0.1689 & -0.0178 & 6.4885 & -3.8031 & -0.3443 & 0.0026 \end{bmatrix},$$

$$G = \begin{bmatrix} 1.2911 & 0.0047 & -0.0554 & -0.1570 \\ 0.0105 & 0.0215 & -0.0100 & 0.0016 \\ 0.0409 & -0.0308 & 7.0044 & -1.2413 \\ -0.0569 & 0.0038 & -1.4706 & 4.4937 \end{bmatrix},$$

and the resultant filter is given by

$$\hat{A} = \begin{bmatrix} -0.2438 & 0.0065 \\ -1.5239 & 0.8483 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} -0.0108 \\ -0.0039 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 0.6628 & -0.0273 \end{bmatrix}.$$

The actual bound versus the uncertain parameter δ is depicted in Figure 4.1. The dash-line is for the improved method and the solid is for the iterative method.

Now we consider the H_∞ filtering problem for the case of $|\delta| \leq 0.45$. Given the initial value $[\lambda_1, \lambda_2, \varepsilon] = [-0.5, -0.5, -0.2]$, we arrive at the minimum value of $\gamma = 2.1558$ with $\lambda_1 = -0.5678$, $\lambda_2 = 0.1245$ and $\varepsilon = 0.0053$ by solving the LMIs in Theorem 4.4.1 and **fminsearch**. Note that the value of γ is much smaller than 3.2065 which

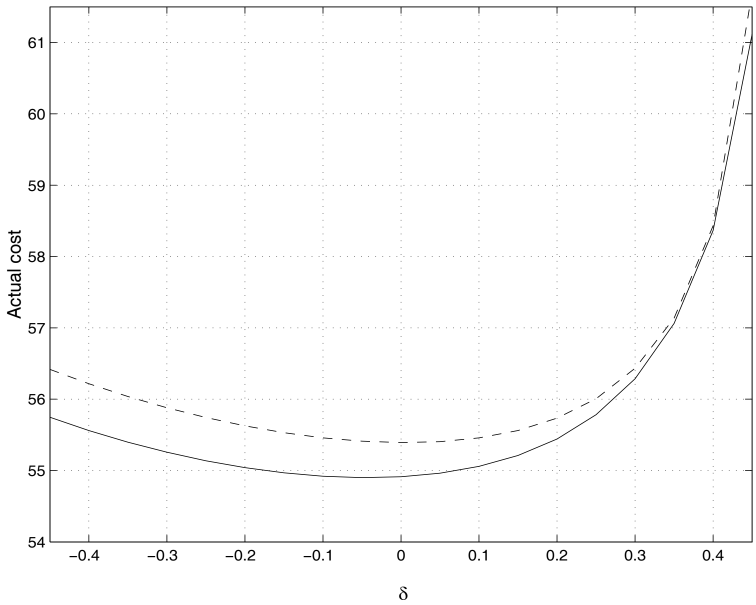


Figure 4.1 Actual bound versus the uncertain parameter for example 4.5.1.

can be obtained by using the method in [SXS01] or [GdOB02].

4.5.2 The example in [SXS01]

Consider the target tracking problem with a discrete state-space model in [SXS01]:

$$x(k+1) = \begin{bmatrix} 0 & -0.8187+\delta \\ 1 & -0.9854+2\delta \end{bmatrix} x(k) + \begin{bmatrix} -6 \\ 1 \end{bmatrix} w(k) \quad (4.5.4)$$

$$y(k) = \begin{bmatrix} -100 & 10 \end{bmatrix} x(k) + v(k) \quad (4.5.5)$$

$$z(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \quad (4.5.6)$$

where the state vector $x(k)$ denotes the coordinate of the target, $w(k)$ and $v(k)$ are uncorrelated zero-mean white noises with unit variances. The parameter δ describes the uncertainty in the turning rate which satisfies $|\delta| \leq \delta_o$, where δ_0 is also a positive real number. Three cases are also considered in this example and the simulation results are shown in Table 4.2.

	$\delta_0 = 0.08$	$\delta_0 = 0.1$	$\delta_0 = 0.12$
First method in [SXS01]	70.40	87.25	111.45
Iterative method in [SXS01]	63.95	77.39	96.80
Proposed Theorem 4.3.1	68.03	82.67	105.80
Proposed iterative method	61.85	72.80	87.74

Table 4.2 Performance comparison among different methods for example 4.5.2.

The results shown in Table 4.2 indicate that the proposed Theorem 4.3.1 and the iterative method of this chapter are less conservative than the first method and the improved iterative method in [SXS01] for all three cases. For the case of $\delta_0 = 0.12$, the minimum bound of 105.80 is obtained by Theorem 4.3.1 for $\lambda_1 = 0.4$ and $\lambda_2 = 0.55$ and the filter parameters are given by

$$\hat{A} = \begin{bmatrix} 0.7751 & -1.0212 \\ 0.6916 & -1.1411 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0.0085 \\ -0.0032 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 0.8094 & -0.0199 \end{bmatrix}.$$

Using the iterative method in this chapter, we can obtain the less conservative bound of 87.74 and the filter is given by

$$\hat{A} = \begin{bmatrix} 0.7751 & -1.0212 \\ 0.6916 & -1.1411 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0.0085 \\ -0.0032 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 0.8360 & -0.0047 \end{bmatrix}.$$

4.5.3 The example in [GdOB02]

Consider the discrete-time system in the form of (4.2.1)-(4.2.3) with [GdOB02]

$$A = \begin{bmatrix} 0.9 & 0.1 + 0.06\alpha \\ 0.01 + 0.05\beta & 0.9 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 0 & 1.414 \end{bmatrix}, \ L_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ L_2 = 0$$

where $|\alpha| \leq 1$ and $|\beta| \leq 1$. This is a two-block structured uncertainty which can be described by a four-vertex ploytope. The value of the H_2 guaranteed cost based upon the method in [GdOB02] is 44.0039. Using Theorem 4.3.1 and Remark 4.3.3, the optimal scaling parameters are obtained as $\lambda_1 = -0.9842$ and $\lambda_2 = -0.9747$ with the initial value $[\lambda_1, \lambda_2] = [-0.5, -0.5]$. The optimal H_2 guaranteed cost is 19.4682. The resultant filter is given by

$$\hat{A} = \begin{bmatrix} 0.0705 & 0.0263 \\ 1.2779 & 0.5492 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0.9114 \\ -0.9972 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 1.2885 & 0.2382 \end{bmatrix}.$$

Based upon the above filter and the iterative algorithm, an even less conservative minimum bound of 15.9759 can be obtained and the filter is given by

$$\hat{A} = \begin{bmatrix} 0.0710 & 0.0262 \\ 1.2764 & 0.5496 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0.9110 \\ -0.9960 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 1.2852 & 0.2433 \end{bmatrix}.$$

It should be noted that only 2 steps are needed to obtain the minimum value of the cost and the corresponding filter. The actual performance of the resultant filter is shown in Figure 4.2. It is clear from the figure that the obtained upper bound is not conservative.

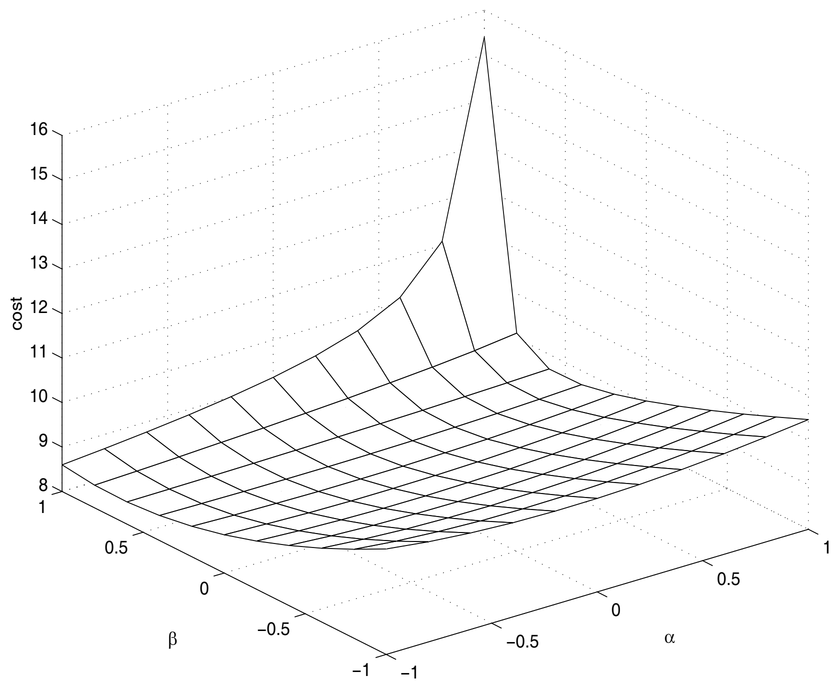


Figure 4.2 Actual bound versus the uncertain parameters α and β.

4.6 Conclusion

This chapter has addressed the robust minimum variance filtering for discrete-time linear systems with polytopic uncertainty. Based on a parameter dependent Lyapunov function approach, we presented less conservative designs of H_2 and H_∞ filters in terms of the improved LMIs than existing approaches. The improved LMIs contain a number of slack variables which offer additional flexibility in optimization. The solution of LMIs, if exists, provides a robust filter with a minimum upper bound to the variance of the filtering error or a guaranteed H_∞ level of noise attenuation over the entire uncertainty polytope. We also proposed an iterative approach to further improve the filter performance. A comparison has been made with existing results.

PART II

NETWORKED CONTROL SYSTEMS

Chapter 5

Robust H_∞ and H_2 Filtering for Discrete-time Systems with Limited Communication

5.1 Introduction

It has been a long time to separate the communication aspects from the dynamics of a system, because this simplifies the problem and usually works well in classic models, just as we have studied in the Part I. However in the situations where the system performance is limited and degraded because of limited communication capacity of a shared network, it is not the same case as before any more.

Conventionally, an estimation problem is tackled by assuming that all the outputs of a system are available to the estimator at any time instant [BC01,SXS01]. This is not the case due to the presence of the network with limited communication capacity. So the limited bandwidth must be taken into consideration when designing an estimator. In [LW96] and [WB97], coding and state estimation in limited communication channels are taken into account and several conditions on the transmission data rate for the existence of convergent coder-estimator are given. The problem of estimating the state of a remote system via a digital communication channel with finite data rate is investigated in [NE97, NE98] and the structures of the optimal coder and estimator are derived. The state estimation problem is studied for discrete time systems with intermittent communication process which is common in distributed systems communicating over a network in [SS02]. A stationary Markov process is used to model probabilistic measurements and a method to obtain the filter gain is also given. A distributed state estimation problem is studied in [SL01] for an interconnected power system in which real-time telemetered data can be exchanged among control centers and the measurements are randomly delayed. An optimal stochastic state estimator is presented which relies on the statistics of the delayed measurements.

In this chapter, we jointly formulate the robust H_∞ or H_2 filtering with limited communication constraint. By using the notion of communication sequence, the networked filtering system can be converted into a periodic system. Then we propose a direct approach to design periodic filters in terms of LMIs. A suboptimal communication sequence and the corresponding optimal filter can be obtained by a combined heuristic search and convex optimization. It should be noted that only in [RS00] the periodic characteristic in this problem is taken into account, while the method used there is based on the lifting technique which has difficulty in handling uncertainty in state space.

5.2 Problem Formulation and Preliminaries

Consider a networked filtering system shown in Figure 5.1. The plant and the filter are spatially distributed and connected together through a control network, whereas the conventional point-to-point link may be used. In the control network,

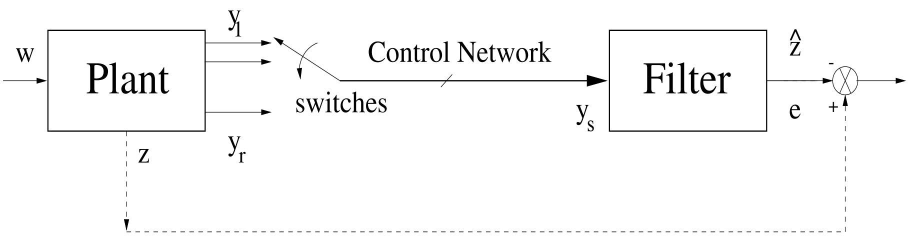


Figure 5.1 Networked Filtering System

there mainly exist two means of transmission: one-packet transmission and multi-packet transmission. One-packet transmission means that all the output signals are lumped together and transmitted in one packet. If we ignore the lumping time and the transmission time in the network, we can view this case as that all the outputs are obtained by the filter at the same time. In multi-packet transmission there are three types of releasing policy: zero releasing policy, random releasing policy and scheduled releasing policy, see [LMT01b] for details. Under the zero releasing policy, every node tries to send its first message at time instant $k = 0$ and send a new message every period. The random releasing policy assumes a random start time for each node and each node still sends a new message every period. Under the scheduled releasing policy, the start-sending time is scheduled to occur for each node and the message is transmitted periodically.

Under the scheduled releasing policy, it is clear that a filter cannot have simultaneous access to all the outputs of the plant, but in a way that the multiple outputs are sequentially multiplexed from the sensors to the filter at every step periodically. The way of multiplexing can be described by the r switches:

$$\sigma_1 = [1 \quad 0 \quad \cdots \quad 0]$$

$$\sigma_2 = [0 \quad 1 \quad \cdots \quad 0]$$

$$\vdots$$

$$\sigma_r = [0 \quad 0 \quad \cdots \quad 1].$$

The switches σ_i, $i = 1, \cdots, r$, are $1 \times r$ matrices with the i-th element being 1 and all other entries being zero, where r is the total number of the outputs. It determines the filter to communicate with which element of the outputs, because $\sigma_i y(k) = y_i(k)$, where $y_i(k)$ is the i-th element of the outputs.

Now we will employ the idea of "communication sequence" which was originally introduced in [Bro95] to jointly formulate the filtering and communication problem. Here the k-th element of a communication sequence, s_k, is defined as an arbitrary element of the above switches. Under the scheduled release policy, the filter communicates with the plant following a periodic pattern, which can be specified by an N-periodic communication sequence s_k, where $s_{k+N} = s_k, \forall k \in \mathcal{Z}$.

Definition 5.2.1 An N-periodic communication sequence s_k, $k = 0, \cdots, N-1$, where $s_k \in \{\sigma_1, \sigma_2, \cdots, \sigma_r\}$, is admissible if the following condition is satisfied:

$$rank \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{bmatrix} = r. \tag{5.2.1}$$

The above definition has a more direct expression and is more flexible in converting a networked filtering system to a periodic system compared with the similar definition given in [HM99]. This condition requires that no more than one of the outputs be received by the filter at each time instant and communication with each of the outputs of the plant at least once within each period be allowed. In this way, the limited communication under the scheduled releasing policy is modelled in a manner that the filter can communicate with only one of the sensors at a discrete-time instant according to the communication sequence.

Now suppose that the plant $(\mathcal{S})$ is a LTI system in the form of (4.2.1)-(4.2.3) in Chapter 4. The plant matrices A, B, C, D, L_1 and L_2 are appropriately dimensioned and they also belong to the uncertainty polytope Ω defined in (4.2.4). Here the system is assumed to be composed of subsystems which are spatially distributed. Each subsystem can only send out its output to the filter by the control network at a given time and is regarded as one node of the networked system.

Since for a given periodic communication sequence, the networked filtering system is also periodic. We introduce a periodic filter of the following form whose period is equal to that of the communication sequence:

$$(\mathcal{F}_k): \ \hat{x}(k+1) = \hat{A}_k \hat{x}(k) + \hat{B}_k y_s(k) \qquad (5.2.2)$$

$$\hat{z}(k) = \hat{H}_k \hat{x}(k) + \hat{J}_k y_s(k) \qquad (5.2.3)$$

where $\hat{x}(k) \in \mathcal{R}^n$ is the state of the filter, $\hat{z}(k) \in \mathcal{R}^q$ is the estimate of signal $z(k)$, $\hat{A}_k \in \mathcal{R}^{n\times n}$, $\hat{B}_k \in \mathcal{R}^{n\times 1}$, $\hat{H}_k \in \mathcal{R}^{q\times n}$, $\hat{J}_k \in \mathcal{R}^{q\times 1}$ are the filter matrices to be determined and they are N-periodic, i.e.,

$$\hat{A}_{k+N} = \hat{A}_k, \quad \hat{B}_{k+N} = \hat{B}_k, \quad \hat{H}_{k+N} = \hat{H}_k, \quad \hat{J}_{k+N} = \hat{J}_k, \quad \forall k \in \mathcal{Z}. \qquad (5.2.4)$$

And $y_s(k)$ is the information which is transmitted from the system $(\mathcal{S})$ and fed into the filter $(\mathcal{F}_k)$ and is defined as

$$y_s(k) = s_k y(k) \qquad (5.2.5)$$

where s_k is the k-th element of the communication sequence. Notice that in general $y_s(k)$ is not the same as $y(k)$ because not all elements of $y(k)$ are communicated with the filter at time instant k.

For convenience, we gather all the filter parameters into the following compact form:

$$\Theta_k = \begin{bmatrix} \hat{A}_k & \hat{B}_k \\ \hat{H}_k & \hat{J}_k \end{bmatrix}. \qquad (5.2.6)$$

Connecting the filter $(\mathcal{F}_k)$ to the plant $(\mathcal{S})$, we have the filtering error system expressed by the following equations:

$$(\mathcal{S}_e):\ \xi(k+1) = A_{s_k}\xi(k) + B_{s_k}w(k) \tag{5.2.7}$$

$$\bar{z}(k) = C_{s_k}\xi(k) + D_{s_k}w(k) \tag{5.2.8}$$

where $\xi(k) = [\, x^T(k) \quad \hat{x}^T(k)\,]^T$, $\bar{z}(k) = z(k) - \hat{z}(k)$ and

$$A_{s_k} = \begin{bmatrix} A & 0 \\ \hat{B}_k s_k C & \hat{A}_k \end{bmatrix},\ B_{s_k} = \begin{bmatrix} B \\ \hat{B}_k s_k D \end{bmatrix},$$
$$C_{s_k} = [\, L_1 - \hat{J}_k s_k C \quad -\hat{H}_k\,],\ D_{s_k} = L_2 - \hat{J}_k s_k D.$$

It is clear that the system $(\mathcal{S}_e)$ is periodic in s_k. The H_∞ or H_2 filtering problem can be stated as follows: find simultaneously a communication sequence s_k and a periodic filter $(\mathcal{F}_k)$ of the form of (5.2.2)-(5.2.3) such that for all uncertainties belonging to the ploytope Ω the filtering error system (5.2.7)-(5.2.8) is asymptotically stable and has an optimal H_∞ or H_2 performance under the scheduled releasing policy.

5.3 Robust H_∞ Filtering

Since for a given periodic communication sequence, the filtering error system is in fact a periodic system, we will first introduce a direct approach for periodic system analysis and design which is in comparison with the traditional lifting technique [RS00].

Consider a linear discrete-time N-periodic system described by the following state space model:

$$(\mathcal{S}_k):\ x(k+1) = A_k x(k) + B_k w(k) \tag{5.3.1}$$

$$z(k) = C_k x(k) + D_k w(k) \tag{5.3.2}$$

where $x(k) \in \mathcal{R}^n$ is the state vector, $w(k) \in \mathcal{R}^p$ is the disturbance input, $z(k) \in \mathcal{R}^q$ is the output of the system, and $A_k \in \mathcal{R}^{n\times n}$, $B_k \in \mathcal{R}^{n\times p}$, $C_k \in \mathcal{R}^{q\times n}$, $D_k \in \mathcal{R}^{q\times p}$ are N-periodic matrices of the system and satisfy

$$A_{k+N} = A_k, \quad B_{k+N} = B_k, \quad C_{k+N} = C_k, \quad D_{k+N} = D_k, \quad \forall k \in \mathcal{Z}. \tag{5.3.3}$$

The transition matrix of the system $(\mathcal{S}_k)$ is defined as

$$\Phi(k,l) = \begin{cases} A_{k-1}A_{k-2}\cdots A_l, & k > l \\ I, & k = l. \end{cases} \tag{5.3.4}$$

The transition matrix $\Phi(k,l)$ is N-periodic in k and l, i.e., $\Phi(k+N, l+N) = \Phi(k,l)$, $\forall k,l \in \mathcal{Z}$. The eigenvalues of $\Phi(k+N,k)$ which are independent of k are referred to as characteristic multipliers. Moreover, the periodic system is stable if and only if all the characteristic multipliers of A_k are inside the unit circle of the complex plane [dST00].

Definition 5.3.1 Given a scalar $\gamma > 0$, the periodic system $(\mathcal{S}_k)$ is said to have an H_∞ performance γ if it is stable and satisfies

$$\sup_{0\neq w\in\ell_2} \frac{\|z\|_2}{\|w\|_2} < \gamma \tag{5.3.5}$$

under zero initial state condition.

Lemma 5.3.1 *[BC01] The N-periodic system $(\mathcal{S}_k)$ has an H_∞ performance γ if and only if either of the following equivalent conditions holds:*

(a) There exists an N-periodic positive definite solution P_k to the periodic Riccati difference inequality

$$\begin{aligned} &A_k^T P_{k+1} A_k - P_k + (A_k^T P_{k+1} B_k + C_k^T D_k)[\gamma^2 I - (B_k^T P_{k+1} B_k + D_k^T D_k)]^{-1} \\ &(A_k^T P_{k+1} B_k + C_k^T D_k)^T + C_k^T C_k < 0 \end{aligned} \tag{5.3.6}$$

such that $\gamma^2 I - (B_k^T P_{k+1} B_k + D_k^T D_k) > 0$.

(b) There exists an N*-periodic positive definite matrix* P_k *satisfying the LMI*

$$\begin{bmatrix} -P_{k+1} & P_{k+1}A_k & P_{k+1}B_k & 0 \\ A_k^T P_{k+1} & -P_k & 0 & C_k^T \\ B_k^T P_{k+1} & 0 & -\gamma I & D_k^T \\ 0 & C_k & D_k & -\gamma I \end{bmatrix} < 0, \quad k = 0, 1, \cdots N-1. \tag{5.3.7}$$

The following lemma can be established by using a similar argument as in [SXS01].

Lemma 5.3.2 *The inequality of (5.3.7) is satisfied if and only if there exist* $P_k \in \mathcal{R}^{n\times n}$ *and* $G_k \in \mathcal{R}^{n\times n}$ *that satisfy*

$$\begin{bmatrix} P_{k+1} - (G_{k+1} + G_{k+1}^T) & 0 & G_{k+1}^T A_k & G_{k+1}^T B_k \\ 0 & -I & C_k & D_k \\ A_k^T G_{k+1} & C_k^T & -P_k & 0 \\ B_k^T G_{k+1} & D_k^T & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{5.3.8}$$

Remark 5.3.1 When setting $G_k = G_k^T = P_k$ and by some row-column exchanges, the inequality (5.3.8) reduces to (5.3.7). However, if the matrices A_k, B_k, C_k, D_k are from an uncertain polytope, (5.3.8) would render a less conservative H_∞ filter design due to the additional slack variable G_k. Note that P_k can be vertex-dependent, which can further reduce the conservatism as compared to (5.3.7) where a fixed Lyapunov function is used.

In order to achieve the optimal H_∞ performance we want to determine a communication sequence and a periodic filter. At the first step, we will obtain a pre-specified noise attenuation level for a given communication sequence while ensuring that the filtering error system (5.2.7)-(5.2.8) is stable.

By Lemma 5.3.2, the filtering error system (5.2.7)-(5.2.8) has an H_∞ performance γ under a given communication s_k if there exist periodic matrices $P_{k+1}^{(i)}$, $P_k^{(i)}$, G_{k+1}

such that

$$\begin{bmatrix} P_{k+1}^{(i)} - (G_{k+1} + G_{k+1}^T) & 0 & G_{k+1}^T A_{s_k}^{(i)} & G_{k+1}^T B_{s_k}^{(i)} \\ 0 & -I & C_{s_k}^{(i)} & D_{s_k}^{(i)} \\ A_{s_k}^{(i)T} G_{k+1} & C_{s_k}^{(i)T} & -P_k^{(i)} & 0 \\ B_{s_k}^{(i)T} G_{k+1} & D_{s_k}^{(i)T} & 0 & -\gamma^2 I \end{bmatrix} < 0 \qquad (5.3.9)$$

for $k = 0, 1, \cdots N-1$ and $i = 1, 2, \cdots, M$, where $A_{s_k}^{(i)}$, $B_{s_k}^{(i)}$, $C_{s_k}^{(i)}$, $D_{s_k}^{(i)}$ are the matrices in (5.2.7)-(5.2.8) with A, B, C, D, L_1, L_2 being at the i-th vertex of the ploytope Ω.

Note that the above matrix inequality (5.3.9) is nonlinear in unknown variables. Thus our purpose is, if possible, to convert (5.3.9) into an LMI. If this goal is accomplished, the H_∞ filter design problem under a given communication sequence turns out to be a convex programming problem which can be solved by very efficient numerical methods [GNLC95]. To this end, we shall use the approach of change of variables as proposed in [GA95] to derive an explicit expression for the filter parameters.

Theorem 5.3.1 *Given a scalar $\gamma > 0$, the H_∞ filter design problem under a given communication sequence s_k for the system (4.2.1)-(4.2.3) is solvable if there exist N-periodic matrices X_k, R_k, T_k, $Q_{1k}^{(i)}$, $Q_{2k}^{(i)}$, $Q_{3k}^{(i)}$, $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{H}_k$ and $\mathcal{J}_k$ satisfying the following set of LMIs:*

$$\begin{bmatrix} -R_{k+1} - R_{k+1}^T + Q_{1(k+1)}^{(i)} & -R_{k+1}^T - X_{k+1} - T_{k+1}^T + Q_{2(k+1)}^{(i)} & 0 \\ * & -X_{k+1} - X_{k+1}^T + Q_{3(k+1)}^{(i)} & 0 \\ * & * & -I \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$$\left.\begin{matrix} R_{k+1}^T A^{(i)} & R_{k+1}^T A^{(i)} & R_{k+1}^T B^{(i)} \\ X_{k+1}^T A^{(i)} + \mathcal{B}_k s_k C^{(i)} + \mathcal{A}_k & X_{k+1}^T A^{(i)} + \mathcal{B}_k s_k C^{(i)} & X_{k+1}^T B^{(i)} + \mathcal{B}_k s_k D^{(i)} \\ L_1^{(i)} - \mathcal{J}_k s_k C^{(i)} - \mathcal{H}_k & L_1^{(i)} - \mathcal{J}_k s_k C^{(i)} & L_2^{(i)} - \mathcal{J}_k s_k D^{(i)} \\ -Q_{1k}^{(i)} & -Q_{2k}^{(i)} & 0 \\ * & -Q_{3k}^{(i)} & 0 \\ * & * & -\gamma^2 I \end{matrix}\right] < 0 \qquad (5.3.10)$$

for $k = 0, 1, \cdots, N-1$ and $i = 1, 2, \cdots, M$ simultaneously. The filter parameters can be obtained by

$$\hat{A}_k = \mathcal{A}_k T_k^{-1} \qquad (5.3.11)$$

$$\hat{B}_k = \mathcal{B}_k \qquad (5.3.12)$$

$$\hat{H}_k = \mathcal{H}_k T_k^{-1} \qquad (5.3.13)$$

$$\hat{J}_k = \mathcal{J}_k. \qquad (5.3.14)$$

Proof Observe from (5.3.9) that G_k is invertible. Denote

$$G_k = \begin{bmatrix} X_k & M_{1k} \\ M_k & U_k \end{bmatrix}, \ G_k^{-1} = \begin{bmatrix} Y_k & N_{1k} \\ N_k & V_k \end{bmatrix}, \ \bar{J}_k = \begin{bmatrix} Y_k & I \\ N_k & 0 \end{bmatrix}. \qquad (5.3.15)$$

We have

$$\bar{J}_{k+1}^T G_{k+1} \bar{J}_{k+1} = \begin{bmatrix} Y_{k+1}^T & Y_{k+1}^T X_{k+1} + N_{k+1}^T M_{k+1} \\ I & X_{k+1} \end{bmatrix} \qquad (5.3.16)$$

$$\bar{J}_{k+1}^T G_{k+1}^T A_{s_k}^{(i)} \bar{J}_k = \begin{bmatrix} A^{(i)} Y_k & A^{(i)} \\ X_{k+1}^T A^{(i)} Y_k + M_{k+1}^T (\hat{B} s_k C^{(i)} Y_k + \hat{A}_k N_k) & X_{k+1}^T A^{(i)} + M_{k+1}^T \hat{B}_k s_k C^{(i)} \end{bmatrix} \qquad (5.3.17), (5.3.18)$$

$$\bar{J}_{k+1}^T G_{k+1}^T B_{s_k}^{(i)} = \begin{bmatrix} B^{(i)} \\ X_{k+1}^T B^{(i)} + M_{k+1}^T \hat{B}_k s_k D^{(i)} \end{bmatrix} \qquad (5.3.19)$$

$$C_{s_k}^{(i)} \bar{J}_k = [\, L_1^{(i)} Y_k - \hat{J}_k s_k C^{(i)} Y_k - \hat{H}_k N_k \quad L_2^{(i)} - \hat{J}_k s_k C^{(i)} \,]. \qquad (5.3.20)$$

Define

$$\bar{J}_k^T P_k^{(i)} \bar{J}_k = \begin{bmatrix} \bar{Q}_{1k}^{(i)} & \bar{Q}_{2k}^{(i)} \\ \bar{Q}_{2k}^{(i)T} & Q_{3k}^{(i)} \end{bmatrix}. \qquad (5.3.21)$$

Pre- and post-multiplying (5.3.9) by $diag\{\bar{J}_{k+1}^T, I, \bar{J}_k^T, I\}$ and $diag\{\bar{J}_{k+1}, I, \bar{J}_k, I\}$, we have

$$\begin{bmatrix} -Y_{k+1}-Y_{k+1}^T+\bar{Q}_{1(k+1)}^{(i)} & -I-Y_{k+1}^TX_{k+1}-N_{k+1}^TM_{k+1}+\bar{Q}_{2(k+1)}^{(i)} & 0 & A^{(i)}Y_k & A^{(i)} & B^{(i)} \\ * & -X_{k+1}-X_{k+1}^T+\bar{Q}_{3(k+1)}^{(i)} & 0 & \Upsilon^{(i)} & X_{k+1}^TA^{(i)}+M_{k+1}^T\hat{B}_ks_kC^{(i)} & X_{k+1}^TB^{(i)}+M_{k+1}^T\hat{B}_ks_kD^{(i)} \\ * & * & -I & L_1^{(i)}Y_k-\hat{J}_ks_kC^{(i)}Y_k-\hat{H}_kN_k & L_1^{(i)}-\hat{J}_ks_kC^{(i)} & L_2^{(i)}-\hat{J}_ks_kD^{(i)} \\ * & * & * & -\bar{Q}_{1k}^{(i)} & -\bar{Q}_{2k}^{(i)} & 0 \\ * & * & * & * & -\bar{Q}_{3k}^{(i)} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{5.3.22}$$

where $\Upsilon^{(i)} = X_{k+1}^TA^{(i)}Y_k + M_{k+1}^T(\hat{B}_ks_kC^{(i)}Y_k + \hat{A}_kN_k)$. From the (1,1)-block of (5.3.22), we know that Y_k is invertible. Denote $Y_k^{-1} = R_k$ and

$$Q_k^{(i)} = \begin{bmatrix} Q_{1k}^{(i)} & Q_{2k}^{(i)} \\ Q_{2k}^{(i)T} & Q_{3k}^{(i)} \end{bmatrix} = \begin{bmatrix} R_k^T\bar{Q}_{1k}^{(i)}R_k & R_k^T\bar{Q}_{2k}^{(i)} \\ \bar{Q}_{2k}^{(i)T}R_k & \bar{Q}_{3k}^{(i)} \end{bmatrix}. \tag{5.3.23}$$

Also define

$$T_k = M_k^TN_kR_k \tag{5.3.24}$$
$$\mathcal{A}_k = M_{k+1}^T\hat{A}_kN_kR_k \tag{5.3.25}$$
$$\mathcal{B}_k = M_{k+1}^T\hat{B}_k \tag{5.3.26}$$
$$\mathcal{H}_k = \hat{H}_kN_kR_k \tag{5.3.27}$$
$$\mathcal{J}_k = \hat{J}_k. \tag{5.3.28}$$

Then (5.3.10) follows by multiplying (5.3.22) from the left and the right by $diag\{R_{k+1}^T,$ $I, I, R_k^T, I , I\}$ and $diag\{R_{k+1}, I ,I, R_k, I , I\}$.

Similarly as in [SXS01], we will show that if the set of LMIs (5.3.10) admits N-periodic solutions X_k, R_k, T_k, $Q_{1k}^{(i)}$, $Q_{2k}^{(i)}$, $Q_{3k}^{(i)}$, then T_k is invertible. First, it follows from (5.3.10) that

$$\begin{bmatrix} Q_{1k}^{(i)} & Q_{2k}^{(i)} \\ Q_{2k}^{(i)T} & Q_{3k}^{(i)} \end{bmatrix} > 0.$$

Multiply (5.3.10) on the left by $diag\{\Pi, I, I, I, I\}$ and on the right by $diag\{\Pi^T$, I, I,I, $I\}$ respectively, where

$$\Pi = \begin{bmatrix} I & -I \\ I & 0 \end{bmatrix}.$$

Then from the (1,1)-block of the resultant LMI, it can be seen that

$$Q_{1(k+1)}^{(i)} - Q_{2(k+1)}^{(i)} - Q_{2(k+1)}^{(i)T} + Q_{3(k+1)}^{(i)} + T_{k+1} + T_{k+1}^T =$$

$$[I \quad -I] \begin{bmatrix} Q_{1(k+1)}^{(i)} & Q_{2(k+1)}^{(i)} \\ Q_{2(k+1)}^{(i)T} & Q_{3(k+1)}^{(i)} \end{bmatrix} \begin{bmatrix} I \\ -I \end{bmatrix} + T_{k+1} + T_{k+1}^T < 0$$

which implies that T_k is invertible. Thus nonsingular matrices M_k and N_k are guaranteed to exist. From (5.3.24)-(5.3.28) and (5.2.2)-(5.2.3), we have

$$\begin{aligned} \zeta(k+1) &= M_{k+1}^{-T}\mathcal{A}_k R_k^{-1} N_k^{-1}\zeta(k) + M_{k+1}^{-T}\mathcal{B}_k y_s(k) \\ \hat{z}(k) &= \mathcal{H}_k R_k^{-1} N_k^{-1}\zeta(k) + J_k y_s(k). \end{aligned}$$

By denoting $\hat{x}(k) = M_k^T\zeta(k)$ and noting from (5.3.24) that $R_k^{-1}N_k^{-1} = T_k^{-1}M_k^T$, we have

$$\begin{aligned} \hat{x}(k+1) &= \mathcal{A}_k T_k^{-1}\hat{x}(k) + \mathcal{B}_k y_s(k) \\ \hat{z}(k) &= \mathcal{H}_k T_k^{-1}\hat{x}(k) + \mathcal{J}_k y_s(k). \end{aligned}$$

Hence the filter parameters of (5.3.11)-(5.3.14) follow.

It is clear that the optimal H_∞ performance relies on both the communication sequence s_k and the periodic filter Θ_k. Let $\gamma_o(s_k, \Theta_k)$ denote the optimal achievable H_∞ filtering performance for a given communication sequence s_k. Then the optimal H_∞ filtering performance under the scheduled releasing policy can be obtained by the following optimization:

$$\begin{array}{ll} \min\limits_{s_k,\Theta_k} & \gamma_o(s_k,\Theta_k) \\ \textit{subject to} & (5.3.10). \end{array}$$

It should be noted that the above problem is very difficult to settle directly since it is in fact a nonconvex optimization with integer constraint. It may be solved by combining an exhaustive search [RS00] with the LMI optimization (5.3.10) which, however, is computationally expensive. To avoid combinatoric searching explosion, we shall propose a heuristic search method which in conjunction with a convex optimization approach gives a simpler solution to the H_∞ filter design problem.

Heuristic Search Method:

- Form the initial sequence $s_0^o = \{\sigma_1, \sigma_2, \cdots, \sigma_r\}$. If the optimal H_∞ performance γ_0^o under this communication sequence satisfies $\gamma_0^o - \gamma^{opt} < \epsilon$, where ϵ is a pre-specified tolerance and γ^{opt} is the optimal H_∞ performance for the system without communication constraint, i.e., $y_s(k) = y(k)$, then s_0^o is the optimal communication sequence. Otherwise, proceed to the next step.

- Step i ($1 \le i < N_m - r$, where N_m is a given maximum period which the period of the desired sequence should not exceed): Assume that the optimal communication sequence obtained at the previous step is s_{i-1}^o, the corresponding optimal H_∞ performance is γ_{i-1}^o and the optimal filter parameter is Θ_{i-1}^o. Removing the latest switch σ_j in the sequence s_{i-1}^o respectively to form a new sequence $s_{i-1}(\sigma_j^\perp), j = 1, 2, \cdots, r$. Calculate the H_∞ optimal performance denoted by $H_\infty(\sigma_j^\perp)$ under the communication sequence $s_{i-1}(\sigma_j^\perp)$ and the filter parameter Θ_{i-1}^o by using Lemma 5.3.2. Compare the r costs, then let $H_\infty(\sigma_{j_o}^\perp)$ denote the worst performance and $\sigma_{j_0}, j_0 \in \{1, 2, \cdots, r\}$ denote the corresponding switch which has been set to zero. σ_{j_0} has the greatest impact on the filtering performance. Hence add an additional sampling σ_{j_0} to s_{i-1}^o to get a new communication sequence s_i^o and then using Theorem 5.3.1 to obtain the optimal performance γ_i^o and the filter parameter Θ_i^o under the new sequence s_i^o.

- Step $i+1$: If $\gamma_i^o - \gamma^{opt} < \epsilon$ or $|\gamma_i^o - \gamma_{i-1}^o| < \epsilon_1$ or $i = N_m - 1 - r$, where ϵ_1 is the pre-specified tolerance, then stop and record the optimal sequence s_i^o and the filter parameter Θ_i^o. Otherwise, let $i = i+1$ and go back to step i.

Remark 5.3.2 From the heuristic search method, we can see that for each period we just need to design the filter parameters once and evaluate the H_∞ performance r times under the resultant filter. This can avoid the combinatoric explosion and thus reduce the computation cost greatly compared to the exhaustive search method [RS00], especially when r and the period of the optimal communication sequence are large. The examples in section 5.5 will show that the heuristic search method is convergent with respect to the period of the communication sequence.

5.4 Robust H_2 Filtering

We will first characterize the H_2 norm of a periodic system by means of LMIs to analyze the H_2 filtering problem. For the N-periodic system (5.3.1)-(5.3.2) with p inputs and q outputs, e_j is the j-th column vector of the $p \times p$ identity matrix, δ_k is the Dirac delta function, $e_j\delta_k$ is an impulse applied to the j-th input and g_{kj} is the corresponding output of the system under the zero initial condition. For each $k = 0, 1, \cdots, N-1$, it is easy to verify that $g_{kj} = \{\underbrace{0, \cdots, 0}_{k-1}, D_k e_j, C_{k+1}B_k e_j, C_{k+2}A_{k+1}B_k e_j, \cdots\}$.

Definition 5.4.1 [XZZ01] For a stable system $(\mathcal{S}_k)$, its H_2 norm is defined as

$$\|\mathcal{S}_k\|_2 = \sqrt{\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=1}^{p}\|g_{kj}\|_2^2}$$

$$= \sqrt{\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=1}^{p}\left[e_j^T D_k^T D_k e_j + \sum_{l=1}^{\infty} e_j^T W^T(k,l)W(k,l)e_j\right]} \quad (5.4.1)$$

where $W(k,l) = C_{k+1}A_{k+l-1}\cdots A_{k+1}B_k$.

This is a commonly used definition of H_2 norm for periodic systems, and is an extension of the well known H_2 norm for linear time-invariant systems. Such a norm can be calculated by the following lemma.

Lemma 5.4.1 *[XZZ01] If the N-periodic system $(\mathcal{S}_k)$ is stable, then its H_2 norm can be obtained in either of the following two ways:*

(a)

$$\|\mathcal{S}_k\|_2 = \sqrt{\frac{1}{N}\sum_{k=0}^{N-1} trace(D_k^T D_k + B_k^T Q_{k+1} B_k)} \quad (5.4.2)$$

where $Q_k = Q_k^T$ and $Q_{k+N} = Q_k$ is the periodic solution of the following Lyapunov equation

$$A_k^T Q_{k+1} A_k - Q_k + C_k^T C_k = 0, \quad k = 0, 1, \cdots, N-1; \quad (5.4.3)$$

(b)

$$\|\mathcal{S}_k\|_2^2 = \min_{(P_{k+1}, W_{k+1})} \frac{1}{N}\sum_{k=0}^{N-1} trace(W_{k+1}) \quad (5.4.4)$$

subject to

$$\begin{bmatrix} -P_{k+1} & P_{k+1}A_k & 0 \\ A_k^T P_{k+1} & -P_k & C_k^T \\ 0 & C_k & -I \end{bmatrix} < 0 \quad (5.4.5)$$

$$\begin{bmatrix} -P_{k+1} & P_{k+1}B_k & 0 \\ B_k^T P_{k+1} & -W_{k+1} & D_k^T \\ 0 & D_k & -I \end{bmatrix} < 0 \quad (5.4.6)$$

simultaneously for $k = 0, 1, \cdots, N-1$, where $P_k^T = P_k$ satisfying $P_N = P_0$.

In the sequel, we want to find an optimal communication sequence and an optimal filter to minimize an upper bound of the H_2 norm of the filtering error system (5.2.7)-(5.2.8) under the scheduled releasing policy. Note that (5.4.5) and (5.4.6) are respectively equivalent to

$$\begin{bmatrix} P_{k+1} - G_{k+1} - G_{k+1}^T & G_{k+1}^T A_k & 0 \\ A_k^T G_{k+1} & -P_k & C_k^T \\ 0 & C_k & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} P_{k+1} - G_{k+1} - G_{k+1}^T & G_{k+1}^T B_k & 0 \\ B_k^T G_{k+1} & -W_{k+1} & D_k^T \\ 0 & D_k & -I \end{bmatrix} < 0.$$

Then with a given communication sequence s_k, the square of the H_2 norm of the error system (5.2.7)-(5.2.8) can be calculated by the optimization

$$\min_{(P_{k+1}^{(i)}, G_{k+1}, W_{k+1})} \frac{1}{N} \sum_{k=0}^{N-1} trace(W_{k+1}) \tag{5.4.7}$$

subject to

$$\begin{bmatrix} P_{k+1}^{(i)} - G_{k+1} - G_{k+1}^T & G_{k+1}^T A_{s_k}^{(i)} & 0 \\ A_{s_k}^{(i)T} G_{k+1} & -P_k^{(i)} & C_{s_k}^{(i)T} \\ 0 & C_{s_k}^{(i)} & -I \end{bmatrix} < 0 \tag{5.4.8}$$

$$\begin{bmatrix} P_{k+1}^{(i)} - G_{k+1} - G_{k+1}^T & G_{k+1}^T B_{s_k}^{(i)} & 0 \\ B_{s_k}^{(i)T} G_{k+1} & -W_{k+1} & D_{s_k}^{(i)T} \\ 0 & D_{s_k}^{(i)} & -I \end{bmatrix} < 0 \tag{5.4.9}$$

simultaneously for $k = 0, 1, \cdots, N-1$, and $i = 1, 2, \cdots, M$, where $A_{s_k}^{(i)}$, $B_{s_k}^{(i)}$, $C_{s_k}^{(i)}$, $D_{s_k}^{(i)}$ are the matrices at the i-th vertex of the ploytope Ω.

Obviously, the above matrix inequalities are nonlinear in unknown variables. Thus similar to the previous section, we can apply a similar linearization method in the previous section.

Theorem 5.4.1 *The optimal H_2 filter design problem under a given communication sequence s_k for the uncertain system (4.2.1)-(4.2.3) can be solved by the optimization:*

$$\min_{(R_k,T_k,X_k,\mathcal{A}_k,\mathcal{B}_k,\mathcal{H}_k,\mathcal{J}_k,Q_{1k}^{(i)},Q_{2k}^{(i)},Q_{3k}^{(i)})} \frac{1}{N}\sum_{k=0}^{N-1} trace(W_{k+1}) \tag{5.4.10}$$

subject to

$$\begin{bmatrix} -R_{k+1}-R_{k+1}^T+Q_{1(k+1)}^{(i)} & -R_{k+1}^T-X_{k+1}-T_{k+1}^T+Q_{2(k+1)}^{(i)} & R_{k+1}^TA^{(i)} & R_{k+1}^TA^{(i)} & 0 \\ * & -X_{k+1}-X_{k+1}^T+Q_{3(k+1)}^{(i)} & X_{k+1}^TA^{(i)}+\mathcal{B}_ks_kC^{(i)}+\mathcal{A}_k & X_{k+1}^TA^{(i)}+\mathcal{B}_ks_kC^{(i)} & 0 \\ * & * & -Q_{1k}^{(i)} & -Q_{2k}^{(i)} & L_1^{(i)T}-C^{(i)T}s_k^T\mathcal{J}_k^T-\mathcal{H}_k^T \\ * & * & * & -Q_{3k}^{(i)} & L_1^{(i)T}-C^{(i)T}s_k^T\mathcal{J}_k^T \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{5.4.11}$$

and

$$\begin{bmatrix} -R_{k+1}-R_{k+1}^T+Q_{1(k+1)}^{(i)} & -R_{k+1}^T-X_{k+1}^T-T_{k+1}^T+Q_{2(k+1)}^{(i)} & R_{k+1}^TB^{(i)} & 0 \\ * & -X_{k+1}-X_{k+1}^T+Q_{3(k+1)}^{(i)} & X_{k+1}^TB^{(i)}+\mathcal{B}_ks_kD^{(i)} & 0 \\ * & * & -W_{k+1} & L_2^{(i)T}-D^{(i)T}s_k^T\mathcal{J}_k^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{5.4.12}$$

simultaneously for $k = 0, 1, \cdots, N-1$ and $i = 1, 2, \cdots, M$. The filter parameters are given by (5.3.11)-(5.3.14).

Proof Using the notations in (5.3.15)-(5.3.21) and pre-multiplying and post-multiplying (5.4.8) by $diag\{\bar{J}_{k+1}^T, \bar{J}_k^T, I\}$ and $diag\{\bar{J}_{k+1}, \bar{J}_k, I\}$ respectively, (5.4.8) is equivalent to

$$\begin{bmatrix} -Y_{k+1} - Y_{k+1}^T + \bar{Q}_{1(k+1)}^{(i)} & -I - Y_{k+1}^T X_{k+1} - N_{k+1}^T M_{k+1} + \bar{Q}_{2(k+1)}^{(i)} & A^{(i)} Y_k & A^{(i)} & 0 \\ * & -X_{k+1} - X_{k+1}^T + \bar{Q}_{3(k+1)}^{(i)} & \Upsilon^{(i)} & X_{k+1}^T A^{(i)} + M_{k+1}^T \hat{B}_k s_k C^{(i)} & 0 \\ * & * & -\bar{Q}_{1k}^{(i)} & -\bar{Q}_{2k}^{(i)} & Y_k^T L_1^{(i)T} - Y_k^T C^{(i)T} s_k^T \mathcal{J}_k^T - N_k^T \hat{H}_k^T \\ * & * & * & -\bar{Q}_{3k}^{(i)} & L_1^{(i)T} - C^{(i)T} s_k^T \mathcal{J}_k^T \\ * & * & * & * & -I \end{bmatrix} < 0 \quad (5.4.13)$$

where $\Upsilon^{(i)} = X_{k+1}^T A^{(i)} Y_k + M_{k+1}^T (\hat{B}_k s_k C^{(i)} Y_k + \hat{A}_k N_k)$. Next, let $R_k = Y_k^{-1}$ and define

$$\mathcal{H}_k = \hat{H}_k N_k R_k. \quad (5.4.14)$$

Using the notations in (5.3.23)-(5.3.26) and (5.4.14), then (5.4.11) follows by multiplying (5.4.13) from the left and from the right by $diag\{R_{k+1}^T, I, R_k^T, I, I\}$ and $diag\{R_{k+1}, I, R_k, I, I\}$, respectively.

Similarly, pre-multiplying and post-multiplying (5.4.9) by $diag\{\bar{J}_{k+1}^T, I, I\}$ and $diag\{\bar{J}_{k+1}, I, I\}$ respectively and using the notations in (5.3.15)-(5.3.21), (5.4.9) is equivalent to

$$\begin{bmatrix} -Y_{k+1} - Y_{k+1}^T + \bar{Q}_{1(k+1)}^{(i)} & -I - Y_{k+1}^T X_{k+1} - N_{k+1}^T M_{k+1} + \bar{Q}_{2(k+1)}^{(i)} \\ * & -X_{k+1} - X_{k+1}^T + \bar{Q}_{3(k+1)}^{(i)} \\ * & * \\ * & * \end{bmatrix}$$

$$\left.\begin{matrix} B^{(i)} & 0 \\ X_{k+1}^{T}B^{(i)} + M_{k+1}^{T}\hat{B}_k s_k D^{(i)} & 0 \\ -W_{k+1} & L_2^{(i)T} - D^{(i)T}s_k^{T}\mathcal{J}_k^{T} \\ * & -I \end{matrix}\right] < 0. \qquad (5.4.15)$$

Multiplying (5.4.15) from the left and from the right by $diag\{R_{k+1}^T, I, I, I\}$ and $diag\{R_{k+1}, I, I, I\}$, respectively and using the notations in (5.3.23)-(5.3.26) and (5.4.14), (5.4.12) is obtained. The filter parameters can be derived in a similar way as the H_∞ filtering.

Let $\|S(s_k, \Theta_k)\|_2$ denote the optimal achievable H_2 norm for a given communication sequence s_k, then the optimal H_2 filtering performance under the scheduled releasing policy can be obtained by the following optimization:

$$\begin{aligned} &\min_{s_k, \Theta_k} && \|S(s_k, \Theta_k)\|_2 \\ &subject\ to && (5.4.11) - (5.4.12). \end{aligned}$$

It is easy to see that the optimal H_2 filter design problem is also a nonconvex optimization with integer constraint, hence the heuristic search method used in the previous section can be employed again to deal with this problem.

5.5 Illustrative Example

Consider an uncertain discrete-time linear system described by

$$x(k+1) = \left[\begin{array}{cc|cc} 0.8 & 0.2+\delta & 0 & 0 \\ 0 & 0 & 0 & 0.2 \\ \hline 0.1 & 0.2+\delta & 0.2 & 0.6 \\ 0 & -0.1 & 0 & 0.3 \end{array}\right] x(k) + \left[\begin{array}{c} 0.3 \\ 0.8 \\ \hline 0 \\ 1.5 \end{array}\right] w(k) \qquad (5.5.1)$$

$$y(k) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] x(k) + \left[\begin{array}{c} 0.1 \\ \hline 0.4 \end{array}\right] w(k) \tag{5.5.2}$$

$$z(k) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \end{array}\right] x(k) + 0.2w(k) \tag{5.5.3}$$

where δ is an uncertain parameter satisfying $|\delta| \leq 0.1$. It is easy to see that the eigenvalues of the first nominal subsystem ($\delta = 0$) are $\{0.8, 0\}$ and the eigenvalues of the second nominal subsystem are $\{0.2, 0.3\}$. The outputs are transmitted to the filter through a control network with the scheduled releasing policy, which can transmit only one measurement within one sampling period. From the previous discussion, we know the switches for this system are

$$\sigma_1 = [\,1 \quad 0\,], \quad \sigma_2 = [\,0 \quad 1\,]. \tag{5.5.4}$$

In the following we will discuss the optimal H_∞ filtering problem under the scheduled releasing policy for the uncertain system (5.5.1)-(5.5.3). First it can be known that the optimal H_∞ performance without communication limitation is 0.6347. With the initial sequence $s_2 = \{\sigma_1, \sigma_2\}$, we obtain the optimal H_∞ performance of 1.7700 and the corresponding filter parameters. Then without changing σ_2, we set $\sigma_1 = [\,0 \quad 0\,]$. Using Lemma 5.3.2 with the filter, the H_∞ cost is known to be 5.1266. On the other hand, without changing σ_1, we set $\sigma_2 = [\,0 \quad 0\,]$, the H_∞ cost is known as 2.2954. Obviously, the first output is more important than the second one to have a better performance. Thus in order to improve the filtering performance, we add an additional sampling to s_2 to get a new sequence $s_3 = \{\sigma_1, \sigma_2, \sigma_1\}$. Proceeding the heuristic search, we obtain the following results shown in Table 5.1. From the results shown above, it can be seen that more communicating with y_1 within a period of time a better H_∞ performance is achieved. This is because the first subsystem is less stable than the second subsystem. Thus allocating more sampling for y_1 will achieve better performance. However it should be noted that even though the measurement of the second subsystem seems to be less important, it cannot be ignored to achieve better performance. For example, if we set $\sigma_1 = [1 \ 0]$ and $\sigma_2 = [1 \ 0]$ the H_∞

period	sequence	H_∞ norm	$H_\infty(\sigma_1^\perp)$	$H_\infty(\sigma_2^\perp)$
$N = 2$	$\{\sigma_1, \sigma_2\}$	1.7700	**5.1266**	2.2954
$N = 3$	$\{\sigma_1, \sigma_2, \sigma_1\}$	1.5472	**22.1307**	1.9682
$N = 4$	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1\}$	1.4199	**16.8359**	1.6473
$N = 5$	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1\}$	1.4114		
$N = 6$	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_1\}$	1.3990		

Table 5.1 The results of using the heuristic search method

performance will be 2.0164.

If we choose the tolerance parameter to be 0.01, the sequence $s^1_{opt} = \{\sigma_1, \sigma_2, \sigma_1, \sigma_1\}$ can be viewed as the optimal sequence for the H_∞ performance. We obtain the same optimal sequence and the corresponding performance value by using the exhaustive search, which will need 4013879 flops in the H_∞ filter design problem. However, in the above heuristic search, only 316510 flops are needed to obtain the desired sequence.

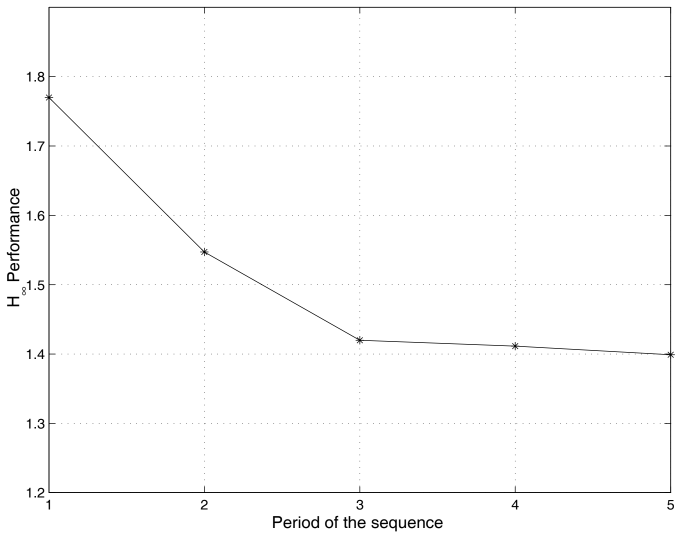


Figure 5.2 Convergence of the H_∞ performance by using the heuristic search method.

The convergence of the heuristic search method for this example is shown in Figure

5.2. The optimal H_∞ filter parameters under the sequence s^1_{opt} are given by

$$\hat{A}_1 = \begin{bmatrix} -0.2893 & -0.0970 & -1.8204 & 17.5391 \\ -0.0425 & -0.0253 & 0.7764 & 0.0709 \\ -0.0350 & 0.0588 & -0.8264 & -2.0813 \\ -0.0025 & 0.0031 & -0.0890 & 0.0227 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} -82.1673 \\ -6.7011 \\ -5.9479 \\ -0.4113 \end{bmatrix},$$

$$\hat{H}_1 = [\,0.0250 \quad 0.0099 \quad -1.3107 \quad 2.5118\,], \quad \hat{J}_1 = 5.1593,$$

$$\hat{A}_2 = \begin{bmatrix} 0.4065 & -0.2198 & -0.8171 & 0.9292 \\ 0.0639 & -0.9806 & 1.7928 & -11.4417 \\ 0.0043 & 0.0499 & 0.3730 & -0.8235 \\ 0.0000 & 0.0038 & 0.0085 & -0.0367 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 1.2387 \\ 4.9548 \\ 0.3582 \\ 0.0036 \end{bmatrix},$$

$$\hat{H}_2 = [\,-0.0290 \quad -0.4804 \quad -0.4264 \quad 0.1172\,], \quad \hat{J}_2 = -0.4519,$$

$$\hat{A}_3 = \begin{bmatrix} -0.7123 & 1.0836 & -16.6125 & 240.7568 \\ 0.9163 & -0.3778 & 8.7771 & -212.8489 \\ 0.0002 & -0.0024 & 0.1042 & -5.5184 \\ 0.0014 & -0.0287 & 0.6923 & -26.9430 \end{bmatrix}, \quad \hat{B}_3 = \begin{bmatrix} -106.1456 \\ 18.6300 \\ -0.0016 \\ 0.0138 \end{bmatrix},$$

$$\hat{H}_3 = [\,0.1189 \quad 0.0132 \quad -0.4355 \quad 20.5052\,], \quad \hat{J}_3 = 3.5056,$$

$$\hat{A}_4 = \begin{bmatrix} -0.6738 & 0.3043 & -7.7986 & -1.5487 \\ 0.2244 & 0.4422 & -2.6366 & -7.4064 \\ -0.0051 & 0.0161 & 0.2282 & -0.3407 \\ -0.0014 & 0.0062 & -0.0007 & -0.1055 \end{bmatrix}, \quad \hat{B}_4 = \begin{bmatrix} -209.8072 \\ 24.6734 \\ -0.6133 \\ -0.1684 \end{bmatrix},$$

$$\hat{H}_4 = [\,0.0442 \quad 0.0008 \quad -1.6773 \quad 0.3951\,], \quad \hat{J}_4 = 6.0385.$$

5.6 Conclusion

This chapter has discussed the H_∞ and H_2 filtering problems for systems in which networks are used to transmit data. The idea of communication sequence was employed to describe the limited bandwidth in the network. Based on the notion of communication sequence, the H_∞ or H_2 filtering was formulated as an optimization problem for periodic systems. For a given communication sequence, a direct LMI based convex optimization approach was developed for the design of an optimal filter. On the other hand, a heuristic search method was proposed to obtain a suboptimal communication sequence. The combined heuristic search and convex optimization approach was demonstrated to give optimal solutions to the simultaneous optimization of the communication sequence and filter, although no theoretical proof has been established.

Chapter 6

Optimal Control of Networked Systems: a Combined Heuristic and Convex Optimization Approach

6.1 Introduction

Control systems with their sensors, actuators and controllers interconnected by communication networks are called networked control systems (NCS), see Figure 6.1. Many researchers have investigated the limited communication constraints when analyzing the system performance or designing controllers for NCS.

D. Hristu addresses the problem of stabilizing an LTI system when only some of its outputs can be measured and/or some of the control actions can be transmitted at one time in [HM99, Hri00], where the so-called "extensification algorithm" is used to transform the stabilization problem to an equivalent problem involving matrix search. The LQ control with communication constraints is studied in [RS00]. It is shown that an optimal communication sequence obtained by exhaustive search for the LQ control is typically such that the sampling resources are focused on where they are needed most. In [NE00a] and [NE00b] the stabilization of infinite-dimensional time-varying ARMA model under limited data rate is considered. It is shown that the optimal finite horizon coder-controller is an optimal quantizer for the initial condition that has been formulated to operate sequentially. In [LW96] and [WB97] coding and state estimation in limited communication channels are taken into account and in [WB99] a new notion of stability is introduced for systems controlled over digital network.

In this chapter, we want to find an optimal communication sequence and a controller to obtain the minimum H_∞ or H_2 cost for networked control systems with limited bandwidth. The problem is first formulated as a periodic control problem by employing the notion of "communication sequence". It is shown that under a given communication sequence, the design of an optimal periodic controller can be converted to a convex optimization. We then propose a heuristic search approach for a suboptimal communication sequence, which together with the convex optimization of the controller, gives a solution to the joint optimization problem. The approach is known to be convergent. As compared to the exhaustive search in [RS00], our approach greatly reduces the computational cost. Several examples are included to demonstrate that the heuristic search in fact converges to a global optimal solution although no theoretical proof is given. It is worth noting that the stabilization problem under limited communication constraint has been considered in [HM99, Hri99] and [IF00] using the lifting technique. A similar method is also used to design a controller to sub-optimize $\mathcal{L}_2$ induced norm for NCS. However, in these works a much higher dimensional system will be dealt with because of using the lifting technique. Our proposed direct approach has the advantage that it can avoid this and can be extended easily to deal with uncertain systems.

6.2 Problem Formulation

The plant, the sensors and the controller in Figure 6.1 are spatially distributed and connected together through a control network. Now suppose that the spatially distributed plant is a linear time-invariant system in the form of (3.2.1)-(3.2.3). The system matrices A, B_1, B_2, C_1, C_2, D_{11}, D_{12} and D_{21} are exactly known with appropriate dimensions. The output of each subsystem can only be sent to the controller through the network at a given time. Here we consider a simple case in which the control u is not transmitted by the network but in a way that it is transmitted to the plant directly.

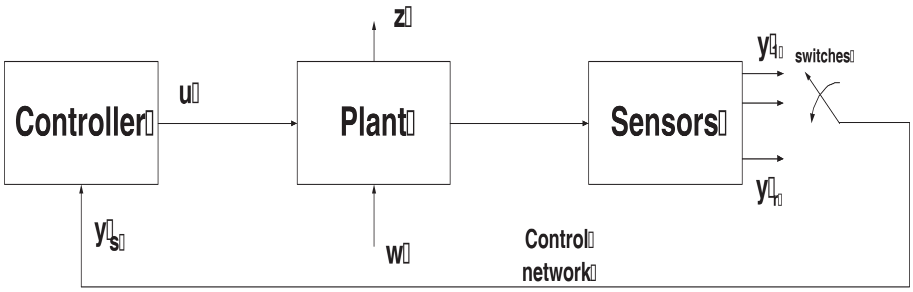


Figure 6.1 Networked Control System

In this chapter, we also assume that the control network adopts the scheduled releasing policy to transmit data. Under this policy the start-sending time is scheduled to occur for each node and the signal is transmitted periodically. From the previous chapter, it is known that under this policy the controller can't have simultaneous access to all the outputs of the plant, but in a way that the multiple outputs are sequentially multiplexed from the sensors to the controller at every step periodically. The way of multiplexing can still be described by the r switches defined in Section 5.1.1. The switch σ_i determines the controller to communicate with which element of the outputs.

We still employ the idea of "communication sequence" which leads the controller to read which of the output signals at each time instant. It is also assumed that the controller communicates with the plant following a periodic pattern, which can be specified by an N-periodic communication sequence s_k, where $s_{k+N} = s_k, \forall k \in \mathcal{Z}$. The sequence is satisfied to be admissible, which requires that no more than one of the outputs be measured by the controller at each time instant and the controller communicate with each of the plant outputs at least once within a period [Hri99]. In this way, the bandwidth limitation in NCS is modeled in a manner that the controller can communicate with only one of the sensors at a discrete-time instant according to the communication sequence.

Remark 6.2.1 If more than one, say l, output measurements of the sensors can be transmitted in one-packet, then $s_k \in \{\bar{\sigma}_0, \bar{\sigma}_1, \cdots, \bar{\sigma}_{N-1}\}$, where

$$\bar{\sigma}_i = \begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \vdots \\ \tilde{\sigma}_l \end{bmatrix}, \ \tilde{\sigma}_i \in \{\sigma_1, \sigma_2, \cdots, \sigma_r\}, \quad i = 0, 1, \cdots, N-1$$

where σ_i is defined as in Section 5.1.1. In this chapter, we mainly consider the case in which no more than one element of the outputs are lumped into one packet. But the results can be extended to the general case easily.

Since for a given periodic communication sequence, the NCS is in fact a periodic system, we introduce a periodic controller of the form of (6.2.1)-(6.2.2) whose period is equal to that of the communication sequence:

$$\hat{x}(k+1) = \hat{A}_k \hat{x}(k) + \hat{B}_k y_s(k) \tag{6.2.1}$$

$$u(k) = \hat{C}_k \hat{x}(k) + \hat{D}_k y_s(k) \tag{6.2.2}$$

where $\hat{x}(k) \in \mathcal{R}^n$ is the state of the controller, $\hat{A}_k \in \mathcal{R}^{n\times n}, \hat{B}_k \in \mathcal{R}^{n\times 1}, \hat{C}_k \in$

$\mathcal{R}^{m\times n}$, $\hat{D}_k \in \mathcal{R}^{m\times 1}$ are the controller matrices which are N-periodic, i.e.,

$$\hat{A}_{k+N} = \hat{A}_k, \quad \hat{B}_{k+N} = \hat{B}_k, \quad \hat{C}_{k+N} = \hat{C}_k, \quad \hat{D}_{k+N} = \hat{D}_k, \quad \forall k \in \mathcal{Z}.$$

For convenience, we gather all the controller parameters into the following compact form:

$$\Theta_k = \begin{bmatrix} \hat{A}_k & \hat{B}_k \\ \hat{C}_k & \hat{D}_k \end{bmatrix}. \tag{6.2.3}$$

And $y_s(k)$ is the information which is transmitted from the plant and fed into the controller. Notice that in general $y_s(k)$ does not equal to $y(k)$ because not all elements of $y(k)$ are communicated to the controller at time instant k. $y_s(k)$ can be described as (5.2.5).

By defining the augmented state vector $\xi(k) = [\, x^T(k) \quad \hat{x}^T(k)\,]^T$, we have the following closed-loop system:

$$(\mathcal{S}_c):\ \xi(k+1) = \bar{A}_{s_k}\xi(k) + \bar{B}_{s_k}w(k) \tag{6.2.4}$$

$$z(k) = \bar{C}_{s_k}\xi(k) + \bar{D}_{s_k}w(k) \tag{6.2.5}$$

where

$$\bar{A}_{s_k} = \begin{bmatrix} A + B_2\hat{D}_k s_k C_2 & B_2\hat{C}_k \\ \hat{B}_k s_k C_2 & \hat{A}_k \end{bmatrix}, \ \bar{B}_{s_k} = \begin{bmatrix} B_1 + B_2\hat{D}_k s_k D_{21} \\ \hat{B}_k s_k D_{21} \end{bmatrix}, \tag{6.2.6}$$

$$\bar{C}_{s_k} = [\, C_1 + D_{12}\hat{D}_k s_k C_2 \quad D_{12}\hat{C}_k\,],\ \bar{D}_{s_k} = D_{11} + D_{12}\hat{D}_k s_k D_{21}\,. \tag{6.2.7}$$

It is clear that the closed-loop system (6.2.4)-(6.2.5) is periodic in k.

The H_∞ or H_2 control problem can be stated as follows: find simultaneously a communication sequence s_k and a periodic controller Θ_k of the form of (6.2.1)-(6.2.2) such that the closed-loop system (6.2.4)-(6.2.5) is asymptotically stable and has an optimal H_∞ or H_2 performance under the scheduled releasing policy.

6.3 H_∞ Control Problem

By Lemma 5.3.1, the closed-loop system (6.2.4)-(6.2.5) has an H_∞ performance γ under a given communication s_k if and only if there exists an N-periodic positive definite matrix X_k such that

$$\begin{bmatrix} -X_{k+1} & X_{k+1}\bar{A}_{s_k} & X_{k+1}\bar{B}_{s_k} & 0 \\ \bar{A}_{s_k}^T X_{k+1} & -X_k & 0 & \bar{C}_{s_k}^T \\ \bar{B}_{s_k}^T X_{k+1} & 0 & -\gamma I & \bar{D}_{s_k}^T \\ 0 & \bar{C}_{s_k} & \bar{D}_{s_k} & -\gamma I \end{bmatrix} < 0, \quad k = 0, 1, \cdots N-1. \tag{6.3.1}$$

Next we shall use the approach of change of variables as proposed in [GA95] to derive an explicit expression for the controller parameters that solve the H_∞ control problem.

Denote

$$X_k = \begin{bmatrix} R_k & M_k \\ M_k^T & U_k \end{bmatrix}, \quad X_k^{-1} = \begin{bmatrix} Y_k & N_k \\ N_k^T & V_k \end{bmatrix} \tag{6.3.2}$$

and

$$\Phi_{2k} = \begin{bmatrix} I & Y_k \\ 0 & N_k^T \end{bmatrix}, \quad \Phi_{1k} = \begin{bmatrix} R_k & I \\ M_k^T & 0 \end{bmatrix}. \tag{6.3.3}$$

Note that the matrices $R_k,\ Y_k,\ M_k,\ N_k,\ U_k$ and V_k are N-periodic. Further it can be easily verified

$$M_k N_k^T = I - R_k Y_k \tag{6.3.4}$$

$$X_k \Phi_{2k} = \Phi_{1k} \tag{6.3.5}$$

$$\Phi_{2k}^T X_k \Phi_{2k} = \begin{bmatrix} R_k & I \\ I & Y_k \end{bmatrix}. \tag{6.3.6}$$

Now introduce the following new controller variables:

$$\mathcal{C}_k = \hat{D}_k s_k C_2 Y_k + \hat{C}_k N_k^T \tag{6.3.7}$$

$$\begin{aligned} \mathcal{B}_k &= R_{k+1}B_2\hat{D}_k + M_{k+1}\hat{B}_k && (6.3.8) \\ \mathcal{A}_k &= R_{k+1}(A + B_2\hat{D}_k s_k C_2)Y_k + R_{k+1}B_2\hat{C}_k N_k^T \\ &\quad + M_{k+1}\hat{B}_k s_k C_2 Y_k + M_{k+1}\hat{A}_k N_k^T. && (6.3.9) \end{aligned}$$

Observe that given matrices $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ and $\hat{D}_k$ and nonsingular matrices M_k and N_k, the controller parameters $\hat{A}_k$, $\hat{B}_k$, $\hat{C}_k$ and $\hat{D}_k$ are uniquely determined by (6.3.7)-(6.3.9).

Theorem 6.3.1 *Given a scalar $\gamma > 0$, the H_∞ control problem under a given communication sequence s_k for the system (3.2.1)-(3.2.3) is solvable if and only if there exist N-periodic matrices $R_k > 0$, $Y_k > 0$, $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ and $\hat{D}_k$ satisfying the following set of LMIs:*

$$\begin{bmatrix} -R_{k+1} & -I & R_{k+1}A + \mathcal{B}_k s_k C_2 & \mathcal{A}_k & R_{k+1}B_1 + \mathcal{B}_k s_k D_{21} & 0 \\ * & -Y_{k+1} & A + B_2\hat{D}_k s_k C_2 & AY_k + B_2\mathcal{C}_k & B_1 + B_2\hat{D}_k s_k D_{21} & 0 \\ * & * & -R_k & -I & 0 & (C_1 + D_{12}\hat{D}_k s_k C_2)^T \\ * & * & * & -Y_k & 0 & (C_1 Y_k + D_{12}\mathcal{C}_k)^T \\ * & * & * & * & -\gamma I & (D_{11} + D_{12}\hat{D}_k s_k D_{21})^T \\ * & * & * & * & * & -\gamma I \end{bmatrix} < 0 \qquad (6.3.10)$$

for $k = 0, 1, \cdots, N-1$, simultaneously.

Proof First, using (6.3.4)-(6.3.6) and the notations in (6.3.7)-(6.3.9), the following identities can be easily derived:

$$\Phi_{2k}^T \bar{C}_{s_k}^T = \begin{bmatrix} (C_1 + D_{12}\hat{D}_k s_k C_2)^T \\ (C_1 Y_k + D_{12}\mathcal{C}_k)^T \end{bmatrix} \tag{6.3.11}$$

$$\Phi_{2(k+1)}^T X_{k+1} \bar{B}_{s_k} = \begin{bmatrix} R_{k+1}B_1 + \mathcal{B}_k s_k D_{21} \\ B_1 + B_2 \hat{D}_k s_k D_{21} \end{bmatrix} \tag{6.3.12}$$

$$\Phi_{2(k+1)}^T X_{k+1} \bar{A}_{s_k} \Phi_{2k} = \begin{bmatrix} R_{k+1}A + \mathcal{B}_k s_k C_2 & \mathcal{A}_k \\ A + B_2 \hat{D}_k s_k C_2 & AY_k + B_2 \mathcal{C}_k \end{bmatrix}. \tag{6.3.13}$$

Then, the LMI (6.3.10) can be obtained easily by pre-multiplying and post-multiplying (6.3.1) by $\mathrm{diag}\{\Phi_{2(k+1)}^T, \Phi_{2k}^T, I, I\}$ and $\mathrm{diag}\{\Phi_{2(k+1)}, \Phi_{2k}, I, I\}$, respectively.

Further, note that if the set of LMIs (6.3.10) admits N-periodic solutions $R_k > 0$, $Y_k > 0$, $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ and $\hat{D}_k$, then it follows from (6.3.10) that

$$\begin{bmatrix} R_k & I \\ I & Y_k \end{bmatrix} > 0.$$

This implies that the matrix $I - R_k Y_k$ is nonsingular and thus nonsingular matrices M_k and N_k that satisfy (6.3.4) are guaranteed to exist. Therefore, the controller parameters can be obtained from (6.3.7)-(6.3.9).

Remark 6.3.1 It is clear that under different communication sequences, the optimal achievable H_∞ performance will be different. Note that the optimal H_∞ performance relies on both the communication sequence and the periodic controller Θ_k. Let $\gamma_o(s_k, \Theta_k)$ be the optimal H_∞ performance under a given communication sequence s_k, then the optimal H_∞ performance under the scheduled releasing policy can be obtained by the following optimization:

$$\begin{aligned} \min_{s_k, \Theta_k} \quad & \gamma_o(s_k, \Theta_k) \\ \textit{subject to} \quad & (6.3.10). \end{aligned}$$

Remark 6.3.2 It should be noted that the above problem is in fact nonconvex with integer constraints, and is very difficult to solve directly. It may be approached by combining an exhaustive search [RS00] with the LMI optimization (6.3.10). However when the number of the measurements increases, the size of the search tree will grow quickly. To avoid combinatoric explosion, in the following, we shall propose a heuristic search method for a communication sequence which, in conjunction with a convex optimization approach for controller parameters, gives a simple solution to the H_∞ control problem.

Heuristic Search Method:

- Form $s_0^o = \{\sigma_1, \sigma_2, \cdots, \sigma_r\}$. If the optimal H_∞ performance γ_0^o under this communication sequence satisfies $\gamma_0^o - \gamma^{opt} < \epsilon$, where ϵ is a pre-specified tolerance and γ^{opt} is the optimal H_∞ performance for the system without communication constraint, i.e., $y_s(k) = y(k)$, then s_0^o is the optimal communication sequence and the period is r. Otherwise, proceed to the next step.

- Step i ($1 \leq i < N_m - r$, where N_m is the maximum period which the period of the desired sequence cannot exceed): Assume that the optimal communication sequence obtained in step $i-1$ is s_{i-1}^o and the optimal cost is γ_{i-1}^o. Add an additional sampling $\sigma_j, j = 1, 2, \cdots, r$, to s_{i-1}^o to form a set of r new switching sequences $s_{ij} = \{s_{i-1}^o, \sigma_j\}, j = 1, 2, \cdots, r$ and the period is increased by one. Calculate the H_∞ optimal performance under these r communication sequences by using Theorem 6.3.1. Assume that the optimal communication sequence among $s_{ij}, j = 1, 2, \cdots, r$, is s_i^o and the optimal cost is γ_i^o.

- Step $i+1$: If $\gamma_i^o - \gamma^{opt} < \epsilon$ or $|\gamma_i^o - \gamma_{i-1}^o| < \epsilon_1$ or $i = N_m - 1 - r$, where ϵ_1 is the pre-specified tolerance, then stop and record the optimal sequence s_i^o and the optimal controller Θ_k^o. Otherwise, let $i = i+1$ and go back to step i.

Remark 6.3.3 From the heuristic search method, we can see that for each period we can just consider r switching sequences. This can avoid the combinatoric explosion and thus reduces the computation cost greatly compared to the exhaustive search method, especially when r and the period of the desired optimal sequence are large. The examples in section 6.5 will show that the heuristic search method is convergent with respect to the period of the communication sequence.

6.4 H_2 Control Problem

By Lemma 5.4.1, the square of the H_2 norm of the closed-loop system (6.2.4)-(6.2.5) under a given communication s_k can be calculated by the optimization:

$$\min_{(X_{k+1},W_{k+1})} \frac{1}{N}\sum_{k=0}^{N-1} trace(W_{k+1}) \tag{6.4.1}$$

subject to

$$\begin{bmatrix} -X_{k+1} & X_{k+1}\bar{A}_{s_k} & 0 \\ \bar{A}_{s_k}^T X_{k+1} & -X_k & \bar{C}_{s_k}^T \\ 0 & \bar{C}_{s_k} & -I \end{bmatrix} < 0 \tag{6.4.2}$$

$$\begin{bmatrix} -X_{k+1} & X_{k+1}\bar{B}_{s_k} & 0 \\ \bar{A}_{s_k}^T X_{k+1} & -W_{k+1} & \bar{D}_{s_k}^T \\ 0 & \bar{D}_{s_k} & -I \end{bmatrix} < 0 \tag{6.4.3}$$

simultaneously for $k = 0, 1, \cdots, N-1$, where $X_k^T = X_k$ satisfying $X_N = X_0$.

Obviously, the above matrices are nonlinear in unknown variables. Thus similar to the H_∞ case, we again use the definition of the new set of variables in (6.3.3) which will convert the problem under consideration to a convex feasibility problem expressed in terms of LMIs. To this end, we also use the partition forms of X_k and its inverse given by (6.3.2) and the variables Φ_{2k} and Φ_{1k} introduced in (6.3.3). In this way, we will get the following theorem.

Theorem 6.4.1 *The optimal H_2 control problem under a given communication se-*

quence s_k for the system (3.2.1)-(3.2.3) can be solved by the following optimization:

$$\min_{(R_k,Y_k,\mathcal{A}_k,\mathcal{B}_k,\mathcal{C}_k,\hat{D}_k)} \frac{1}{N}\sum_{k=0}^{N-1} trace(W_{k+1}) \qquad (6.4.4)$$

subject to

$$\begin{bmatrix} -R_{k+1} & -I & R_{k+1}A+\mathcal{B}_k s_k C_2 & \mathcal{A}_k & 0 \\ * & -Y_{k+1} & A+B_2\hat{D}_k s_k C_2 & AY_k+B_2\mathcal{C}_k & 0 \\ * & * & -R_k & -I & (C_1+D_{12}\hat{D}_k s_k C_2)^T \\ * & * & * & -Y_k & (C_1Y_k+D_{12}\mathcal{C}_k)^T \\ * & * & * & * & -I \end{bmatrix} < 0 \quad (6.4.5)$$

and

$$\begin{bmatrix} -R_{k+1} & -I & R_{k+1}B_1+\mathcal{B}_k s_k D_{21} & 0 \\ * & -Y_{k+1} & B_1+B_2\hat{D}_k s_k D_{21} & 0 \\ * & * & -W_{k+1} & (D_{11}+D_{12}\hat{D}_k s_k D_{21})^T \\ * & * & * & -I \end{bmatrix} < 0 \qquad (6.4.6)$$

simultaneously for $k = 0, 1, \cdots, N-1$, $R_N = R_0$ and $Y_N = Y_0$. The controller parameters are given by (6.3.7)-(6.3.9).

Proof Using the notations in (6.3.7)-(6.3.9), the LMI (6.4.5) can be obtained by pre-multiplying and post-multiplying (6.4.2) by $diag\{\Phi_{2(k+1)}^T, \Phi_{2k}^T, I\}$ and $diag$ $\{\Phi_{2(k+1)}, \Phi_{2k}, I\}$ respectively. Similarly, pre-multiplying and post-multiplying (6.4.3) by $diag\{\Phi_{2(k+1)}^T, I, I\}$ and $diag\{\Phi_{2(k+1)}, I, I\}$ respectively, then using the notations in (6.3.7)-(6.3.9), the LMI (6.4.6) can be obtained.

Remark 6.4.1 Let $\|\mathcal{S}_c(s_k,\Theta_k)\|_2$ denote the optimal achievable H_2 norm for a given communication sequence s_k, then the optimal H_2 control performance under the scheduled releasing policy can be obtained by the following optimization:

$$\begin{aligned} \min_{s_k,\Theta_k} \quad & \|\mathcal{S}_c(s_k,\Theta_k)\|_2 \\ \textit{subject to} \quad & (6.4.5)-(6.4.6). \end{aligned}$$

Note that the H_2 performance $\|\mathcal{S}_c(s_k, \Theta_k)\|_2$ is a nonlinear function of s_k and the controller Θ_k. Similar to the H_∞ case, the H_2 optimal control problem under communication constraint is also a nonconvex optimization with integer constraint, and is very difficult to solve using any direct optimization method. Hence we can use the heuristic search method described in the previous section again to deal with this problem.

6.5 Illustrative Examples

6.5.1 Example 6.1

Consider a discrete-time linear system described by

$$x(k+1) = \left[\begin{array}{cc|cc} 1.45 & 0.2 & 0 & 0 \\ 0 & 0.4 & 0 & 0.2 \\ \hline 1 & 0.2 & 1.1 & 0.75 \\ 0 & -1 & 0 & 0.4 \end{array}\right] x(k) + \left[\begin{array}{c} 0.6 \\ 0 \\ \hline 0.6 \\ 0 \end{array}\right] w(k) + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 1 \end{array}\right] u(k) \quad (6.5.1)$$

$$z(k) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \end{array}\right] x(k) + w(k) + u(k) \quad (6.5.2)$$

$$y(k) = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] x(k) + \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right] w(k) \quad (6.5.3)$$

It can be easily seen that the eigenvalues of the first subsystem are $\{1.45, 0.4\}$ and the eigenvalues of the second subsystem are $\{1.1, 0.4\}$. The feedback from the sensor to the controller is connected by a network with the scheduled releasing policy, which can transmit only one measurement within one sampling period. From the previous section, we know that the switches for this system are

$$\sigma_1 = [1 \quad 0], \quad \sigma_2 = [0 \quad 1].$$

In the following we will discuss the problem of optimal H_∞ and H_2 control under the scheduled releasing policy for the system (6.5.1)-(6.5.3). First it can be known that the optimal H_∞ performance and H_2 performance without communication limitations are 3.9368 and 2.7254. We start from the sequence $\{\sigma_1, \sigma_2\}$ and obtain the optimal H_∞ performance of 4.8842 and the H_2 performance of 3.3383. Then by adding the switch σ_1 and the switch σ_2 to the sequence $\{\sigma_1, \sigma_2\}$ respectively, we arrive at the sequences $\{\sigma_1, \sigma_2, \sigma_1\}$ and $\{\sigma_1, \sigma_2, \sigma_2\}$. By Theorem 6.3.1, we can obtain the H_∞ performance of 4.3947 under the sequence $\{\sigma_1, \sigma_2, \sigma_1\}$ which is smaller than that under the sequence $\{\sigma_1, \sigma_2, \sigma_2\}$ as shown in Table 6.1. So the sequence $\{\sigma_1, \sigma_2, \sigma_1\}$ is the resulted sequence at this step. Proceeding the heuristic search we have the following results shown in Table 6.1 and Table 6.2, in which the sequence in boldface stands for the resultant sequence at each step.

step	*period*	*sequence*	H_∞ *norm*	*sequence*	H_∞ *norm*
$i=1$	$N=3$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1}\}$	**4.3947**	$\{\sigma_1, \sigma_2, \sigma_2\}$	5.6995
$i=2$	$N=4$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1, \sigma_1}\}$	**4.1390**	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_2\}$	5.1950
$i=3$	$N=5$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1}\}$	**3.9647**	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_2\}$	4.8717
$i=4$	$N=6$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_1}\}$	**3.9368**	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_2\}$	4.6338

Table 6.1 The results of using the heuristic search method for H_∞ control problem.

step	*period*	*sequence*	H_2 *norm*	*sequence*	H_2 *norm*
$i=1$	$N=3$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1}\}$	**2.9906**	$\{\sigma_1, \sigma_2, \sigma_2\}$	3.7493
$i=2$	$N=4$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1, \sigma_1}\}$	**2.8313**	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_2\}$	3.3506
$i=3$	$N=5$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1}\}$	**2.7387**	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_2\}$	3.1360
$i=4$	$N=6$	$\{\boldsymbol{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_1}\}$	**2.7254**	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_2\}$	3.0006

Table 6.2 The results of using the heuristic search method for H_2 control problem.

From the results shown above, we can see that under different communication sequences with the same period the optimal H_∞ performances or H_2 performances are different and the more sampling we allocate for $y_1(k)$ the better performance is achieved. This may be explained by the fact that the first subsystem is less stable than the second one. As a result, communicating more often with y_1 will lead to better performance. Clearly, when we choose the communication sequence

$s^1_{opt} = \{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_1\}$, we can achieve the optimal H_∞ or H_2 performance which are in fact the same as those without limited communication. However it should be noted that even though the measurement of the second subsystem seems to play a less important role, it cannot be ignored. In fact, there does not exist any controller to stabilize the system by only using the measurement of the first subsystem.

We can also obtain the optimal sequence and the corresponding performance by using the exhaustive search, which will need 651691 flops in the H_∞ control problem and 735597 flops in the H_2 control problem. However, in the above heuristic search, only 355796 flops and 392651 flops are needed respectively to get the desired sequence. When the number of the outputs of the original system and the periods of the desired sequence become larger, more time will be saved by using the heuristic search method. The convergence of the heuristic search method with respect to the period of the communication sequence for this example is shown in Figure 6.2.

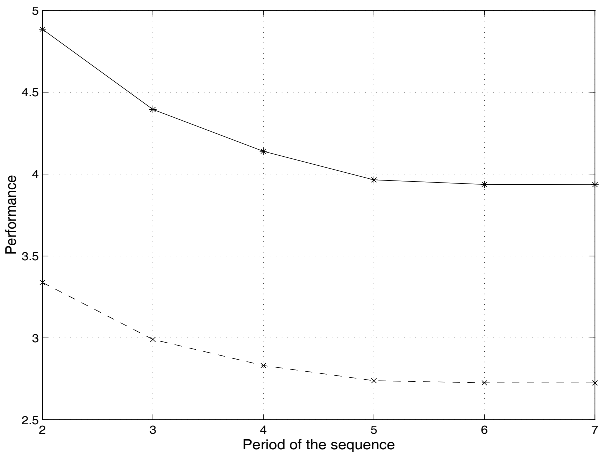


Figure 6.2 Convergence of the heuristic search method: the solid is for H_∞ norm and the dashed is for H_2 norm.

In the simulation, we also tried to get different systems by changing the value of the element $A(1,1)$ from 1.3 to 1.45 and using the heuristic search method to deal with these systems. Here we give four typical cases and the results are shown in the following table.

$A(1,1)$	*optimal sequence*	H_∞ *norm*	$flops_e$	$flops_h$	H_2 *norm*	$flops_e$	$flops_h$
1.43	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1, \sigma_1\}$	3.8559	381642	248268	2.6335	403492	258743
1.4	$\{\sigma_1, \sigma_2, \sigma_1, \sigma_1\}$	3.7336	189656	151772	2.4947	197050	156421
1.38	$\{\sigma_1, \sigma_2, \sigma_1\}$	3.6508	75254	75254	2.4014	75151	75151
1.3	$\{\sigma_1, \sigma_2\}$	3.3106	13924	13924	2.0202	17941	17941

Table 6.3 The optimal performances and sequences for different systems: $flops_e$ stands for the case of exhaustive search and $flops_h$ stands for the case of heuristic search.

It should be noted that the value of H_∞ norm or H_2 norm for each case is exactly the same as that obtained when there is no communication constraint. It can be clearly seen from Table 6.3 that the period of the optimal communication sequence varies with the characteristic of the system.

6.5.2 Example 6.2

To further illustrate the effectiveness of the heuristic search method, in the following we will consider another example with three outputs whose parameters are given by

$$A = \begin{bmatrix} 1.5 & 0.3 & 0.1 & 0 & -0.4 \\ 0 & 0.4 & 0 & 0.2 & -0.1 \\ 0.1 & 0.2 & 0.2 & -0.6 & 0 \\ 0 & -0.1 & 0 & 0.4 & 0.3 \\ 0.3 & -0.5 & 0.6 & 0.3 & -0.4 \end{bmatrix}, B_1 = \begin{bmatrix} 0.6 \\ 0 \\ 0.6 \\ 0 \\ -0.3 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0.6 \end{bmatrix}, C_1^T = \begin{bmatrix} 0.2 \\ 0 \\ 0 \\ 0.8 \\ 0.3 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0.3 \\ 0 & 0 & 1 & 0 & 0.5 \\ 0.1 & 0 & 0.7 & 0 & 0.6 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 1 \\ 0.6 \\ 0 \end{bmatrix}, \quad D_{11} = 0.4, \quad D_{12} = -0.1.$$

The optimal H_∞ norm without communication constraint for this example is 1.2128. By using the heuristic search method we can obtain the desired sequence $s_{opt}^3 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_1, \sigma_1\}$ under which the H_∞ norm is known as 1.6450. This sequence can be viewed as the optimal sequence since the simulation shows that no performance is improved when enlarging the period of the sequence. The optimal H_2 norm without communication constraint is 0.9596 and using the heuristic search method we arrive at the following results: 1.0914 under the sequence $\{\sigma_1, \sigma_2, \sigma_3, \sigma_1, \sigma_1\}$ and 1.0666 under the sequence $\{\sigma_1, \sigma_2, \sigma_3, \sigma_1, \sigma_1, \sigma_1\}$. If we choose the tolerance parameter to be 0.1, the sequence s_{opt}^3 can also be viewed as the optimal sequence for H_2 performance. The convergence of the heuristic search method for this example is shown in Figure 6.3.

6.5.3 Example 6.3

We will consider the following system borrowed from Example 1 in [dOGB02]:

$$A = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ D_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be known that the optimal H_∞ performance without communication constraint is 1.6151. Using the heuristic search method with one step search, we can easily obtain the value of 1.8244 for the H_∞ performance. And following the same procedure, we have the results shown in Figure 6.4. It can be seen that the heuristic search method is convergent and when the period of the sequence is larger than 2, the performance improves little. So the sequence $s^2_{opt} = \{\sigma_1, \sigma_2\}$ can be viewed as the optimal sequence for this example.

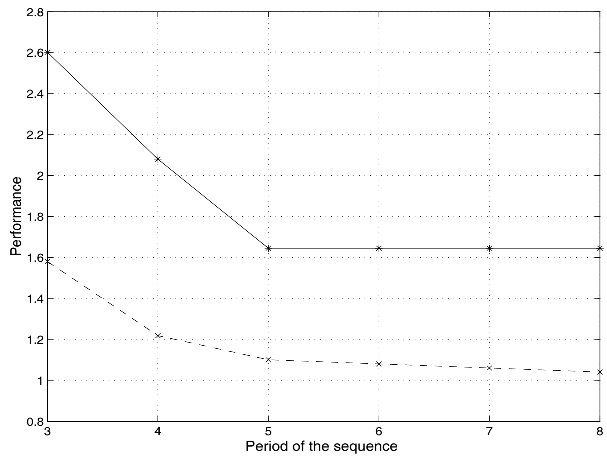


Figure 6.3 Convergence of the heuristic search method: the solid is for H_∞ norm and the dashed is for H_2 norm.

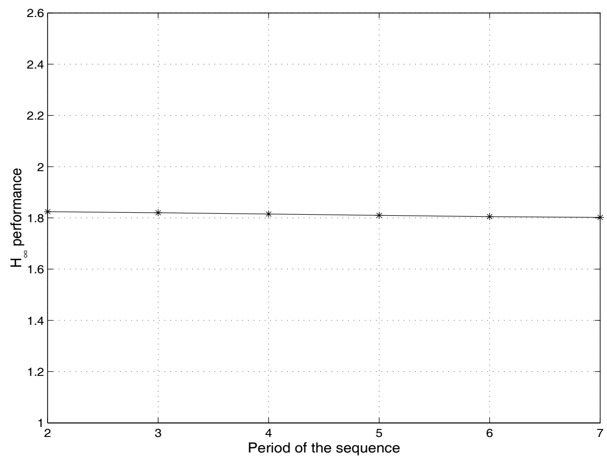


Figure 6.4 Convergence of the heuristic search method.

The H_∞ controller under the sequence s^2_{opt} is given as follows:

$$\hat{A}_1 = \begin{bmatrix} 0.6536 & 1.4917 & -2.7440 & 517.5768 \\ -0.0083 & 0.7378 & 0.6453 & 6.4421 \\ 0.0023 & -0.1053 & 0.9872 & -14.2333 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} 3.6880 \\ 0.0590 \\ -0.0878 \\ -0.0000 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} 0.0477 & -0.1108 & 0.8971 & -217.6819 \\ 0.0181 & -0.0053 & 0.1559 & -82.6930 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} -1.5510 \\ -0.5918 \end{bmatrix}.$$

$$\hat{A}_2 = \begin{bmatrix} 0.6648 & 2.3291 & -0.7537 & -709.0439 \\ -0.0074 & 0.7451 & 0.6726 & 10.2185 \\ 0.0023 & -0.1276 & 0.9360 & 3.8971 \\ -0.0000 & 0.0001 & 0.0002 & 2.4173 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 3.6633 \\ 0.0365 \\ -0.0981 \\ 0.0006 \end{bmatrix},$$

$$\hat{C}_2 = \begin{bmatrix} 0.0425 & -0.4627 & 0.0618 & 416.9544 \\ 0.0161 & -0.1394 & -0.1624 & 107.7530 \end{bmatrix}, \quad \hat{D}_2 = \begin{bmatrix} -1.5511 \\ -0.5910 \end{bmatrix}.$$

6.5.4 Example 6.4: Inverted Pendulum System

The continuous inverted pendulum system model is given as follows [XHH00]:

$$\begin{aligned} \dot{x}_p(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -B_f/M & -3m_2g/4M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3B_f/2LM & 3(m_1+m_2)g/2LM & 0 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -3/2LM \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_p(t) \end{aligned}$$

where $x_p^T(t) = [x \quad \dot{x} \quad \theta \quad \dot{\theta}]^T$ and $m_1 = 1,\ B_f = 0,\ m_2 = 0.5,\ L = 1,\ g = 9.8,\ M = m1 + m_2/4$. Assuming a zero-order sample-and-hold at the inputs with a sampling period equal to 0.1, we can obtain the corresponding discretized system. To deal with the H_2 control problem we also assume that $D_{21} = 0,\ D_{11} = 0,\ D_{12} = 0$ and

$$B_1 = [\,0.2 \quad -0.2 \quad 0.4 \quad -0.0\,]^T,\ C_1 = [\,0.6 \quad 0 \quad 0 \quad 0.3\,].$$

The optimal H_2 performance without communication constraint for the discrete-time system is 1.3267. We start from the sequence $\{\sigma_1, \sigma_2\}$ under which the H_2 performance is calculated as 2.8962. The simulation results via the heuristic search method are given in Table 6.4 and the convergence of this method with respect to the period of the sequence for the inverted pendulum system is shown in Figure 6.5.

step	*period*	*sequence*	H_2 *norm*	*sequence*	H_2 *norm*
$i=1$	$N=3$	$\{\sigma_1,\sigma_2,\sigma_1\}$	3.7297	$\{\boldsymbol{\sigma_1},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2}\}$	**2.7552**
$i=2$	$N=4$	$\{\sigma_1,\sigma_2,\sigma_2,\sigma_1\}$	3.4148	$\{\boldsymbol{\sigma_1},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2}\}$	**2.6755**
$i=3$	$N=5$	$\{\sigma_1,\sigma_2,\sigma_2,\sigma_2,\sigma_1\}$	3.2223	$\{\boldsymbol{\sigma_1},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2}\}$	**2.6232**
$i=4$	$N=6$	$\{\sigma_1,\sigma_2,\sigma_2,\sigma_2,\sigma_2,\sigma_1\}$	3.1142	$\{\boldsymbol{\sigma_1},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2}\}$	**2.5859**
$i=5$	$N=7$	$\{\sigma_1,\sigma_2,\sigma_2,\sigma_2,\sigma_2,\sigma_2,\sigma_1\}$	3.0154	$\{\boldsymbol{\sigma_1},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2},\boldsymbol{\sigma_2}\}$	**2.5579**

Table 6.4 The results of using the heuristic search method for H_2 control problem.

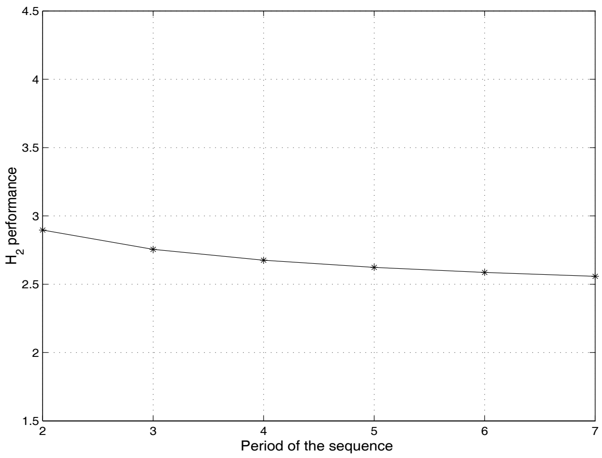


Figure 6.5 Convergence of the heuristic search method.

From the results we can see that allocating more resources to the angle between the stick and the vertical axis is essential to achieve better performance and when the period of the sequence exceeds 5 the performance is improved very little. So the sequence $s^4_{opt} = \{\sigma_1, \sigma_2, \sigma_2, \sigma_2, \sigma_2\}$ may be regarded as the optimal sequence for this inverted pendulum system. Under the sequence s^4_{opt} we can obtain the following controller:

$$\hat{A}_1 = \begin{bmatrix} 1.4295 & 4.4826 & 14.3213 & 30.6403 \\ 0.9741 & 4.4049 & 19.5476 & 36.3217 \\ 0.1826 & 0.0038 & -1.9211 & 11.8135 \\ -0.0000 & -0.0000 & -0.0000 & -0.1659 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} -64.2333 \\ -82.0351 \\ -15.3770 \\ 0.0000 \end{bmatrix},$$

$$\hat{C}_1 = [\,-0.2436 \quad -1.7950 \quad -6.1463 \quad -15.4440\,], \quad \hat{D}_1 = [\,28.6484\,],$$

$$\hat{A}_2 = \begin{bmatrix} 4.2817 & -3.2983 & -2.0543 & 10.4318 \\ 3.3749 & -2.8520 & -2.4912 & 9.4519 \\ -0.2440 & 0.3430 & 0.9059 & -1.5025 \\ 0.0000 & -0.0000 & 0.0000 & 0.0008 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} -282.4769 \\ -279.2660 \\ 20.1874 \\ -0.0000 \end{bmatrix},$$

$$\hat{C}_2 = [\,-1.2481 \quad 1.2679 \quad 0.7601 \quad -3.9912\,], \quad \hat{D}_2 = [\,112.7934\,],$$

$$\hat{A}_3 = \begin{bmatrix} 3.9554 & -4.1886 & -4.0229 & 10.7594 \\ 3.1990 & -3.6711 & -4.4284 & 10.1028 \\ -0.1218 & 0.2722 & 1.0835 & -1.3296 \\ 0.0000 & -0.0000 & 0.0000 & -0.0040 \end{bmatrix}, \quad \hat{B}_3 = \begin{bmatrix} -282.8581 \\ -289.5060 \\ 11.0252 \\ -0.0000 \end{bmatrix},$$

$$\hat{C}_3 = [\,-1.0177 \quad 1.4563 \quad 1.3740 \quad -3.7220\,], \quad \hat{D}_3 = [\,101.6118\,],$$

$$\hat{A}_4 = \begin{bmatrix} 4.0177 & -4.0289 & -3.8659 & 10.4950 \\ 3.3581 & -3.5826 & -4.3284 & 10.0302 \\ -0.0894 & 0.1836 & 1.0288 & -1.1596 \\ 0.0000 & -0.0000 & 0.0000 & -0.0046 \end{bmatrix}, \quad \hat{B}_4 = \begin{bmatrix} -302.5802 \\ -317.7872 \\ 8.4619 \\ -0.0000 \end{bmatrix},$$

$$\hat{C}_4 = [\,-0.9710 \quad 1.2993 \quad 1.2371 \quad -3.3769\,], \quad \hat{D}_4 = [\,101.3939\,],$$

$$\hat{A}_5 = \begin{bmatrix} 3.8811 & -3.7294 & -3.0390 & 8.1365 \\ -1.3586 & 1.4331 & 2.2977 & -3.8885 \\ 0.1916 & -0.2613 & -1.1867 & 1.3691 \\ -0.0000 & 0.0000 & -0.0000 & 0.0180 \end{bmatrix}, \quad \hat{B}_5 = \begin{bmatrix} -301.4499 \\ 132.8850 \\ -18.7442 \\ 0.0001 \end{bmatrix},$$

$$\hat{C}_5 = [\,-0.9374 \quad 1.1950 \quad 0.9926 \quad -2.6187\,], \quad \hat{D}_5 = [\,101.1956\,].$$

6.6 Conclusion

In this chapter we have investigated the optimal H_∞ and H_2 control problems for networked control systems. Based on the notion of the communication sequence and a direct controller design approach for periodic systems, the solutions to the problems are given in terms of a set of LMIs. Given a sequence, an explicit expression for the optimal controller was provided in terms of the solution of the LMIs. It has been shown that communication sequences can affect the system performance. A heuristic search method was given to obtain the desired sequence.

Chapter 7

H_2 Controller Design for Networked Control Systems

7.1 Introduction

In the previous two chapters, we have mainly focused on networked systems from the point of view of communication constraint. It is known that the utilization of a network may result in random communication delays and data loss in the feedback loop. The delays and the data loss will deteriorate the system performance or even make the system unstable. The effect of communication time delay in NCS on system performance has been investigated by many researchers [Ray94, NBW98, LMT01a]. From this chapter, we will study the networked control systems (NCS) by focusing on the effect of time delay on system stability and performance. A block diagram of NCS is illustrated again in Figure 7.1, where h^{sc} denotes the delay from the sensor to the controller and h^{ca} denotes the delay from the controller to the actuator.

Because of the variability of network induced delays, the NCS will be time-varying in general which makes the analysis and control design difficult. When an upper

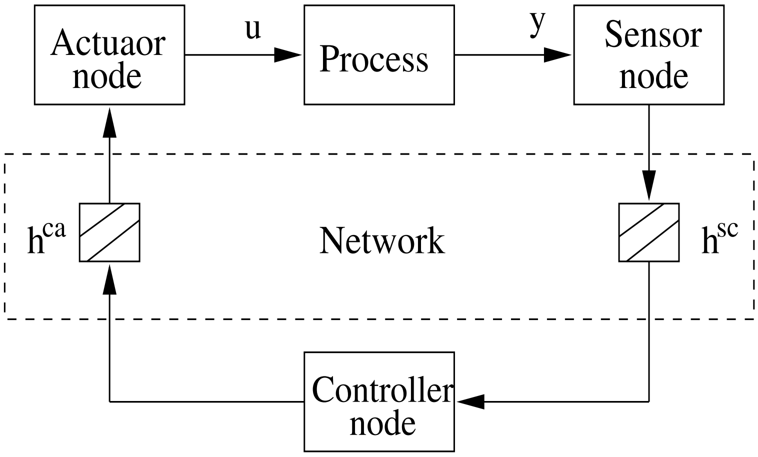


Figure 7.1 Networked Control System.

bound for the communication delay is known, one way to deal with the problem is to augment the state to include the past delayed signals. In [RLS93, Ray94, NBW98], only the case where the combined delay from the sensor to the controller and from the controller to the actuator is less than one sampling period is considered.

Analysis and modeling of an NCS with communication delays are investigated in [LMT01a, KOC+94], where the stability and performance of the closed-loop system are studied for a controller which is designed without considering the impact of communication delays. From the point of view of stochastic stability, a controller design method is proposed in [XHH00], where an NCS is modeled as a finite dimensional, discrete-time jump linear system, with the transition jumps in the form of finite-state Markov chains. A controller is then designed based on system augmentation and a V-K iterative process for solving bilinear matrix inequalities (BMIs). An optimal state feedback controller is presented in [LMT02] to compensate for the multiple time delays. In [KKP02], a dynamic output feedback controller is given to stabilize an asymmetric networked control system, where controller parameters can be obtained by efficiently solving the LMIs based upon the deterministic switched system theory.

In this chapter, we shall assume that upper bounds for the network induced delays from the sensor to the controller and from the controller to the actuator are known. Through output and input augmentation, the dynamic output feedback control problem is converted to a static output feedback one with time-varying parameter uncertainties. Therefore the networked control problem is reformulated as a switched static output feedback control problem. As compared to [XHH00, KOC+94], where the state augmentation is employed, we adopt the augmentation of output and/or input, which will lead to a much lower system dimension. We derive conditions for the existence of an H_2 controller in terms of bilinear matrix inequality(BMI), and extend the so-called sequential linear programming matrix method (SLPMM) [Lei01] for finding a suboptimal fixed order controller. The SLPMM algorithm is presented in [Lei01] to deal with the BMI problem and is known to be convergent. We shall extend the algorithm to handle BMIs with multiple bilinear terms in our problem.

7.2 Preliminary

Consider a stable LTI system described by the following state space equation:

$$(\mathcal{S}): \; x(k+1) = Ax(k) + Bw(k) \tag{7.2.1}$$

$$z(k) = Cx(k) + Dw(k) \tag{7.2.2}$$

where $x(k) \in \mathcal{R}^n$ is the state vector, $w(k) \in \mathcal{R}^p$ is the disturbance input, $z(k) \in \mathcal{R}^q$ is the output of the system, and A, B, C, D are constant matrices.

It is well known that the H_2 norm can be evaluated through controllability and observability Gramians as

$$\|\mathcal{S}\|_2^2 = trace(CQC^T + DD^T) = trace(B^T PB + D^T D) \tag{7.2.3}$$

where the controllability gramian P and the observability gramian Q are the solutions to the following Lyapunov equations:

$$AQA^T - Q + BB^T = 0 \tag{7.2.4}$$

$$A^T PA - P + C^T C = 0 \tag{7.2.5}$$

respectively.

The H_2 norm can also be obtained by the following LMI optimization method.

Lemma 7.2.1 *[XSD02]*

$$\|\mathcal{S}\|_2^2 = \min_{\Sigma,P} trace(\Sigma) \tag{7.2.6}$$

subject to

$$\begin{bmatrix} -P & PA & 0 \\ A^T P & -P & C^T \\ 0 & C & -I \end{bmatrix} < 0 \tag{7.2.7}$$

$$\begin{bmatrix} -P & PB & 0 \\ B^T P & -\Sigma & D^T \\ 0 & D & -I \end{bmatrix} < 0 \tag{7.2.8}$$

simultaneously with $P^T = P$ *and* $\Sigma^T = \Sigma$.

Lemma 7.2.2 *[GA94] Given a symmetric* $\Phi \in \mathcal{R}^{m\times m}$ *and two matrices* P *and* Q *of column dimension* m*, consider the problem of finding some matrix* Θ *of compatible dimensions such that*

$$\Phi + P^T \Theta^T Q + Q^T \Theta P < 0. \tag{7.2.9}$$

Then (7.2.9) is solvable for Θ *if and only if*

$$W_P^T \Phi W_P < 0, \quad W_Q^T \Phi W_Q < 0 \tag{7.2.10}$$

where W_p, W_Q *denote any matrices whose columns form bases of the null space of* P *and* Q*, respectively.*

7.3 The Case of Delay from Sensor to Controller

In this chapter, a discrete-time LTI system in the form of (3.2.1)-(3.2.3) will be considered. The measurement $y(k)$ is transmitted from the sensor to the controller with random but bounded delays τ^{sc}, where $1 \leq \underline{\tau}^{sc} \leq \tau^{sc} \leq \bar{\tau}^{sc} < \infty$. Then the input to the controller is $y_s(k) = y(k - \tau^{sc})$.

Since the delay is time-varying, we introduce a time-varying controller with state space realization of the form:

$$\hat{x}(k+1) = \hat{A}_k \hat{x}(k) + \hat{B}_k y_s(k) \tag{7.3.1}$$

$$u(k) = \hat{C}_k \hat{x}(k) + \hat{D}_k y_s(k) \tag{7.3.2}$$

where $\hat{x}(k) \in \mathcal{R}^{\hat{n}}$ is the state of the controller and $\hat{A}_k,\ \hat{B}_k,\ \hat{C}_k,\ \hat{D}_k$ are the controller matrices to be determined. Note that the controller order is not necessarily equal to the plant order.

Denoting $\rho(k+1) = C_2 x(k) + D_{21} w(k)$ and $x_\rho(k) = [\,\hat{x}^T(k) \quad x^T(k) \quad \rho^T(k) \quad \rho^T(k-1) \quad \cdots \quad \rho^T(k - \bar{\tau}^{sc} + 1)\,]^T$, we have the following closed-loop system:

$$x_\rho(k+1) = \begin{bmatrix} \hat{A}_k & 0 & [& \hat{B}_k W_i & &] \\ B_2\hat{C}_k & A & [& B_2\hat{D}_k W_i & &] \\ 0 & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} x_\rho(k) + \begin{bmatrix} 0 \\ B_1 \\ D_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} w(k) \tag{7.3.3}$$

$$z(k) = [\,D_{12}\hat{C}_k \quad C_1 \quad D_{12}\hat{D}_k W_i\,]\, x_\rho(k) + D_{11} w(k). \tag{7.3.4}$$

The matrix W_i is defined as follows:

$$W_i = [\,\underbrace{0 \ \cdots \ I \ \cdots \ 0}_{\bar{\tau}^{sc} - \underline{\tau}^{sc} + 1 \text{ terms}} \ \underbrace{0 \ \cdots \ 0}_{\underline{\tau}^{sc} \text{ terms}}\,], \quad \tau^{sc} \in [\underline{\tau}^{sc},\ \bar{\tau}^{sc}]. \tag{7.3.5}$$

Note that W_i has all elements being zero except for the τ^{sc}-th block being an identity matrix, where $i = 1, 2, \cdots, \bar{\tau}^{sc} - \underline{\tau}^{sc} + 1$. For example, if $\bar{\tau}^{sc} = 2$ and $\underline{\tau}^{sc} = 1$, we have $W_1 = [\,0 \quad I \quad 0\,]$ and $W_2 = [\,I \quad 0 \quad 0\,]$.

By gathering all the controller parameters into the following compact form

$$\Theta_k = \begin{bmatrix} \hat{A}_k & \hat{B}_k \\ \hat{C}_k & \hat{D}_k \end{bmatrix}, \tag{7.3.6}$$

we can rewrite the system (7.3.3)-(7.3.4) as follows:

$$x_\rho(k+1) = (A_{0\rho} + B_\rho \Theta_k C_{\rho,i}) x_\rho(k) + B_{0\rho} w(k) \tag{7.3.7}$$

$$z(k) = (C_{0\rho} + D_\rho \Theta_k C_{\rho,i}) x_\rho(k) + D_{11} w(k) \tag{7.3.8}$$

where $i = 1, 2, \cdots, \bar{\tau}^{sc} - \underline{\tau}^{sc} + 1$ and

$$A_{0\rho} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & 0 & \cdots & 0 & 0 \\ 0 & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}, \ B_\rho = \begin{bmatrix} I & 0 \\ 0 & B_2 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \ B_{0\rho} = \begin{bmatrix} 0 \\ B_1 \\ D_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C_{0\rho} = [\,0 \quad C_1 \quad 0\,],\ D_\rho = [\,0 \quad D_{12}\,],\ C_{\rho,i} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & W_i \end{bmatrix}.$$

From the system (7.3.7) and (7.3.8), we can see that the dynamic output feedback control problem is converted to a static output feedback one.

Suppose that the sensor data is time stamped. At time instant k, if no new sensor output is received, the output at $k-1$ will be used as the controller input. Hence, in this case the delay $\tau_k^{sc} = \tau_{k-1}^{sc} + 1$. On the other hand, if at time instant k, several sensor outputs are received, the most recent measurement will be used while discarding the other measurements. Therefore, the delay τ_k^{sc} satisfies

$$1 \leq \tau_k^{sc} \leq \tau_{k-1}^{sc} + 1. \tag{7.3.9}$$

Then, the system (7.3.7)-(7.3.8) can be viewed as a switched system with the switching rule as

$$i \to j, \ \ j = 1, \cdots, i+1.$$

Hence, a switching controller can be employed, say Θ_i.

In view of Lemma 7.2.1 and the switched system theory [DRI02], an upper bound of the square of the H_2 norm of the system (7.3.7)-(7.3.8) can be calculated by

$$\min_{P_i, \Sigma, \Theta_i} trace(\Sigma)$$

subject to

$$\begin{bmatrix} -P_j + B_{0\rho}B_{0\rho}^T & (A_{0\rho} + B_\rho \Theta_i C_{\rho,i}) \\ (A_{0\rho} + B_\rho \Theta_i C_{\rho,i})^T & -P_i^{-1} \end{bmatrix} < 0 \tag{7.3.10}$$

$$\begin{bmatrix} -\Sigma + D_{11}D_{11}^T & (C_{0\rho} + D_\rho \Theta_i C_{\rho,i}) \\ (C_{0\rho} + D_\rho \Theta_i C_{\rho,i})^T & -P_i^{-1} \end{bmatrix} < 0 \tag{7.3.11}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc} - \underline{\tau}^{sc} + 1$; $j = i+1$. By letting $P_i^{-1} = Q_i$ in (7.3.10) and (7.3.11), the square of the H_2 norm can be obtained by

$$\min_{P_i, Q_i, \Theta_i, \Sigma,} trace(\Sigma)$$

subject to

$$\begin{bmatrix} -P_j + B_{0\rho}B_{0\rho}^T & (A_{0\rho} + B_\rho \Theta_i C_{\rho,i}) \\ (A_{0\rho} + B_\rho \Theta_i C_{\rho,i})^T & -Q_i \end{bmatrix} < 0 \tag{7.3.12}$$

$$\begin{bmatrix} -\Sigma + D_{11}D_{11}^T & (C_{0\rho} + D_\rho \Theta_i C_{\rho,i}) \\ (C_{0\rho} + D_\rho \Theta_i C_{\rho,i})^T & -Q_i \end{bmatrix} < 0 \tag{7.3.13}$$

$$P_i Q_i = I \tag{7.3.14}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc} - \underline{\tau}^{sc} + 1$; $j = i + 1$. (7.3.12) and (7.3.13) can be rewritten as

$$\Psi_i + M_1^T \Theta_i N_i + N_i^T \Theta_i^T M_1 \quad < \quad 0 \tag{7.3.15}$$

$$\Phi_i + M_2^T \Theta_i N_i + N_i^T \Theta_i^T M_2 \quad < \quad 0 \tag{7.3.16}$$

where

$$\Psi_i = \begin{bmatrix} -P_j + B_{0\rho} B_{0\rho}^T & A_{0\rho} \\ A_{0\rho}^T & -Q_i \end{bmatrix} \tag{7.3.17}$$

$$\Phi_i = \begin{bmatrix} -\Sigma + D_{11} D_{11}^T & C_{0\rho} \\ C_{0\rho}^T & -Q_i \end{bmatrix} \tag{7.3.18}$$

$$M_1 = [\, B_\rho^T \quad 0 \,] \tag{7.3.19}$$

$$M_2 = [\, D_\rho^T \quad 0 \,] \tag{7.3.20}$$

$$N_i = [\, 0 \quad C_{\rho,i} \,] \tag{7.3.21}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc} - \underline{\tau}^{sc} + 1$; $j = i + 1$. By Lemma 7.2.2, the inequalities (7.3.15) and (7.3.16) hold for some Θ_i if and only if the matrix inequalities

$$M_1^{\perp T} \Psi_i M_1^{\perp} < 0, \qquad N_i^{\perp T} \Psi_i N_i^{\perp} < 0 \tag{7.3.22}$$

$$M_2^{\perp T} \Phi_i M_2^{\perp} < 0, \qquad N_i^{\perp T} \Phi_i N_i^{\perp} < 0 \tag{7.3.23}$$

are satisfied, where $M_1^{\perp}$, $M_2^{\perp}$ and $N_i^{\perp}$ are the bases of Ker(M_1), Ker(M_2) and Ker(N_i), respectively. The equation (7.3.14) can be weakened to the following well known semidefinite programming relaxation:

$$\min \sum_{i=1}^{N_\rho} trace(P_i Q_i) \tag{7.3.24}$$

subject to

$$\mathcal{K}(P_i, Q_i) = \begin{bmatrix} -P_i & I \\ I & -Q_i \end{bmatrix} \le 0 \tag{7.3.25}$$

where $N_\rho = \bar{\tau}^{sc} - \underline{\tau}^{sc} + 1$. The condition $P_i Q_i = I$ is satisfied if and only if the optimal value of $trace(P_i Q_i)$ equals to n_x, where n_x is the dimension of P_i [Lei01].

The above problem is not convex since the function $trace(P_i Q_i)$ is bilinear. This bilinear problem has been investigated in Chapter 3 by using the so-called SLPMM method. Similar to the procedure in Chapter 3, we extend the SLPMM method again to solve the robust static H_2 output feedback control and have the following result.

Theorem 7.3.1 *Consider the system (7.3.3)-(7.3.4) with a known upper bound $\bar{\tau}^{sc}$ on communication delay from the sensor to the controller. Then, a solution to the H_2 optimal output feedback controller can be obtained by the following optimization algorithm:*

- *Step 1: Obtain the initial values (P_i^0, Q_i^0, Σ^0) satisfying (7.3.22), (7.3.23), (7.3.25).*
- *Step 2: Given (P_i^k, Q_i^k), obtain a solution of (P_i, Q_i, Σ), denoted by $(M_i^k$, N_i^k, $Z^k)$, to the convex optimization:*

$$\begin{aligned} \min_{P_i, Q_i, \Theta_i, \Sigma} \quad & \left(\sum_{i=1}^{N_\rho} trace(P_i Q_i^k + P_i^k Q_i) + trace(\Sigma) \right) \\ \textit{subject to} \quad & (7.3.22), (7.3.23), (7.3.25). \end{aligned}$$

- *Step 3: If* $\left| \sum_{i=1}^{N_\rho} trace(M_i^k Q_i^k + P_i^k N_i^k) + trace(Z^k) - 2\sum_{i=1}^{N_\rho} trace(P_i^k Q_i^k) - trace(\Sigma^k) \right| \leq \epsilon$*, stop, where ϵ is a pre-defined sufficiently small positive scalar.*
- *Step 4: Compute $\alpha \in [0,1]$ by solving*

$$\min_{\alpha \in [0,1]} \sum_{i=1}^{N_\rho} \quad trace((P_i^k + \alpha(M_i^k - P_i^k))(Q_i^k + \alpha(N_i^k - Q_i^k)))$$

$$+trace(\Sigma^k + \alpha(Z^k - \Sigma^k)).$$

- *Step 5: Set* $P_i^{k+1} = (1-\alpha)P_i^k + \alpha M_i^k$, $Q_i^{k+1} = (1-\alpha)Q_i^k + \alpha N_i^k$, $\Sigma^{k+1} = (1-\alpha)\Sigma^k + \alpha Z^k$, *go to step 2.*

Remark 7.3.1 Suppose the above optimization leads to positive definite solutions P_i and Q_i, $i = 1, 2, \cdots, \bar{\tau}^{sc} - \underline{\tau}^{sc} + 1$. Then, with the derived P_i and Q_i, we can obtain the optimal H_2 controller parameters Θ_i by solving the LMIs (7.3.12) and (7.3.13).

7.4 The Case of Delay from Controller to Actuator

In this section, we consider the case in which there is no delay from the sensor to the controller but there exist delays from the controller to the actuator. That is, the connection from the controller to the plant is via a communication network whereas the controller is directly connected to the plant. Since the controller cannot obtain the information of the time delay from the controller to the actuator, a switching control cannot be applied. Hence, the controller of the following form is employed:

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k) \\ u_c(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k) \end{aligned}$$

where $u_c(k)$ is the output of the controller. Suppose the delay from the controller to the actuator is τ^{ca} which is also assumed to be random but bounded. Then the control input of the plant can be expressed as

$$u(k) = u_c(k - \tau^{ca})$$

where $1 \leq \underline{\tau}^{ca} \leq \tau^{ca} \leq \bar{\tau}^{ca} < \infty$.

Denoting $\eta(k+1) = u_c(k) = \hat{C}\hat{x}(k) + \hat{D}y(k)$ and $x_\eta(k) = [\,\hat{x}^T(k) \quad x^T(k) \quad \eta^T(k) \quad \eta^T(k-1) \cdots \quad \eta^T(k-\bar{\tau}^{ca}+1)\,]^T$, we have the following closed-loop system:

$$
\begin{aligned}
x_\eta(k+1) &= \begin{bmatrix} \hat{A} & \hat{B}C_2 & 0 & \cdots & 0 & 0 \\ 0 & A & [& B_2U_i & &] \\ \hat{C} & \hat{D}C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} x_\eta(k) + \begin{bmatrix} \hat{B}D_{21} \\ B_1 \\ \hat{D}D_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} w(k) \quad (7.4.1) \\
z(k) &= [\,0 \quad C_1 \quad D_{12}U_i\,]\, x_\eta(k) + D_{11}w(k) \quad (7.4.2)
\end{aligned}
$$

where the matrix U_i is defined as follows:

$$
U_i = [\,\underbrace{0 \ \cdots \ 0}_{\underline{\tau}^{ca}-1 \text{ terms}} \ \underbrace{0 \ \cdots \ I \ \cdots \ 0}_{\bar{\tau}^{ca}-\underline{\tau}^{ca}+1 \text{ terms}}\,], \quad \tau^{ca} \in [\underline{\tau}^{ca}, \ \bar{\tau}^{ca}]. \quad (7.4.3)
$$

That is, U_i has all elements being zero except the τ^{ca}-th block which is an identity matrix, where $i = \underline{\tau}^{ca}, \underline{\tau}^{ca}+1, \cdots, \bar{\tau}^{ca}$. We can rewrite the closed-loop system (7.4.1)-(7.4.2) into the following form:

$$
\begin{aligned}
x_\eta(k+1) &= (A_{0\eta,i} + B_\eta \Theta C_\eta^1)x_\eta(k) + (B_{0\eta} + B_\eta \Theta C_\eta^2)w(k) \quad (7.4.4) \\
z(k) &= C_{0\eta,i}x_\eta(k) + D_{11}w(k) \quad (7.4.5)
\end{aligned}
$$

where $i = \underline{\tau}^{ca}, \underline{\tau}^{ca}+1, \cdots, \bar{\tau}^{ca}$ and

$$
A_{0\eta,i} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & [& B_2U_i & &] \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}, \ B_\eta = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \ B_{0\eta} = \begin{bmatrix} 0 \\ B_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
$$

$$C_\eta^1 = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & C_2 & 0 & \cdots & 0 & 0 \end{bmatrix},\ C_\eta^2 = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix},\ C_{0\eta,i} = [\,0 \quad C_1 \quad D_{12}U_i\,].$$

Similar to the previous section, we can easily obtain the following theorem.

Theorem 7.4.1 *Consider the system (7.4.4)-(7.4.5) with a known upper bound, $\bar{\tau}^{ca}$, on communication delay from the controller to the actuator. A solution to the H_2 optimal dynamic output feedback control can be obtained by solving the following optimization:*

$$\min_{\Sigma, P_i, Q_i, \Theta} \Big(\sum_{i=\underline{\tau}^{ca}}^{\bar{\tau}^{ca}} trace(P_i Q_i) + trace(\Sigma) \Big) \tag{7.4.6}$$

subject to

$$\begin{bmatrix} -Q_i & A_{0\eta,i} + B_\eta \Theta C_\eta^1 \\ (A_{0\eta,i} + B_\eta \Theta C_\eta^1)^T & -P_j + C_{0\eta,i}^T C_{0\eta,i} \end{bmatrix} < 0 \tag{7.4.7}$$

$$\begin{bmatrix} -Q_i & B_{0\eta} + B_\eta \Theta C_\eta^2 \\ (B_{0\eta} + B_\eta \Theta C_\eta^2)^T & -\Sigma + D_{11}^T D_{11} \end{bmatrix} < 0 \tag{7.4.8}$$

$$\begin{bmatrix} -P_i & I \\ I & -Q_i \end{bmatrix} \leq 0$$

for $i, j = \underline{\tau}^{ca}, \underline{\tau}^{ca}+1, \cdots, \bar{\tau}^{ca}$.

It should be noted that for the case of time delay from the controller to the actuator, delay is unknown. Hence j has to be from $\underline{\tau}^{ca}$ to $\bar{\tau}^{ca}$.

Remark 7.4.1 Obviously, the term $trace(P_i Q_i)$ in the equation (7.4.6) is also bi-linear. Similar to the previous section, it is easy to eliminate the controller matrix Θ in (7.4.7) and (7.4.8) by using Lemma 7.2.2. Hence, an iterative algorithm similar to that in Theorem 7.3.1 can be used to obtain an optimal solution to the H_2 control problem.

7.5 The case of Combined Delays

In this section, we will consider the case in which there are communication delays from the sensor to the controller and from the controller to the actuator. Since the sensor measurement is time-stamped, a switching controller can still be applied. On the other hand, the time delay from the controller to the actuator can't be known to the controller. Hence a combined switching control and robust control of the last two sections can be applied to design a controller. However, for the convenience of expression, we will use a fixed controller described by

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}y_s(k) \tag{7.5.1}$$

$$u_c(k) = \hat{C}\hat{x}(k) + \hat{D}y_s(k). \tag{7.5.2}$$

By denoting $\zeta(k) = u_c(k) = \hat{C}\hat{x}(k) + \hat{D}y_s(k)$ and $\rho(k+1) = y_s(k) = C_2x(k) + D_{21}w(k)$, the following closed-loop system can be obtained

$$\chi(k+1) = \begin{bmatrix} \hat{A} & 0 & \hat{B}W_i & 0 & 0 \\ 0 & A & 0 & [& B_2U_j] \\ 0 & \tilde{C}_2 & \tilde{I}_1 & 0 & 0 \\ \hat{C} & 0 & \hat{D}W_i & 0 & 0 \\ 0 & 0 & 0 & \tilde{I}_2 & \tilde{I}_3 \end{bmatrix} \chi(k) + \begin{bmatrix} 0 \\ B_1 \\ D_{21} \\ 0 \\ 0 \end{bmatrix} w(k) \tag{7.5.3}$$

$$z(k) = [\,0 \quad C_1 \quad 0 \quad D_{12}U_j\,]\,\chi(k) + D_{11}w(k) \tag{7.5.4}$$

where

$$\chi(k) = \begin{bmatrix} \hat{x}(k) \\ x(k) \\ \bar{\rho}(k) \\ \zeta(k) \\ \bar{\zeta}(k) \end{bmatrix}, \quad \bar{\rho}(k) = \begin{bmatrix} \rho(k) \\ \rho(k-1) \\ \vdots \\ \rho(k-\bar{\tau}^{sc}+2) \\ \rho(k-\bar{\tau}^{sc}+1) \end{bmatrix}, \quad \bar{\zeta}(k) = \begin{bmatrix} \zeta(k-1) \\ \zeta(k-2) \\ \vdots \\ \zeta(k-\bar{\tau}^{ca}+2) \\ \zeta(k-\bar{\tau}^{ca}+1) \end{bmatrix},$$

$$\tilde{C}_2 = \begin{bmatrix} C_2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{I}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad \tilde{I}_2 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

and $\tilde{I}_3$ has the same structure as $\tilde{I}_1$ but with different dimension. The matrices W_i and U_j are defined as (7.3.5) and (7.4.3). The system (7.5.3)-(7.5.4) can be expressed by

$$\chi(k+1) = (A_{0\zeta,j} + B_\zeta \Theta C_{\zeta,i})\chi(k) + B_{0\zeta} w(k) \quad (7.5.5)$$

$$z(k) = C_{0\zeta,j}\chi(k) + D_{11} w(k) \quad (7.5.6)$$

where

$$A_{0\zeta,j} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & [& B_2 U_j] \\ 0 & \tilde{C}_2 & \tilde{I}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{I}_2 & \tilde{I}_3 \end{bmatrix}, \quad B_\zeta = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad B_{0\zeta} = \begin{bmatrix} 0 \\ B_1 \\ D_{21} \\ 0 \\ 0 \end{bmatrix},$$

$$C_{\zeta,i} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & W_i & 0 & 0 \end{bmatrix}, \quad C_{0\zeta,j} = [\, 0 \quad C_1 \quad 0 \quad D_{12} U_j \,].$$

Theorem 7.5.1 *Consider the system (7.5.5)-(7.5.6) with known upper bounds, $\bar{\tau}^{sc}$ and $\bar{\tau}^{ca}$, on communication delays from the sensor to the controller and from the controller to the actuator. Then, a solution to the H_2 optimal dynamic output feedback control problem can be obtained by solving the following optimization:*

$$\min_{\Sigma, P_{i,j}, Q_{i,j}, \Theta} \Big(\sum_{i=1}^{N_\rho} \sum_{j=\underline{\tau}^{ca}}^{\bar{\tau}^{ca}} trace(P_{i,j} Q_{i,j}) + trace(\Sigma) \Big) \quad (7.5.7)$$

subject to

$$\begin{bmatrix} -Q_{i,j} & A_{0\zeta,j} + B_{\zeta}\Theta C_{\zeta,i} \\ (A_{0\zeta,j} + B_{\zeta}\Theta C_{\zeta,i})^T & -P_k + C_{0\zeta,j}^T C_{0\zeta,j} \end{bmatrix} < 0$$

$$\begin{bmatrix} -Q_{i,j} & B_{0\zeta} \\ B_{0\zeta}^T & -\Sigma + D_{11}^T D_{11} \end{bmatrix} < 0$$

$$\begin{bmatrix} -P_{i,j} & I \\ I & -Q_{i,j} \end{bmatrix} \leq 0$$

for $i \in \mathcal{I} = \{1, 2, \cdots, N_\rho\}$, $j \in \mathcal{J} = \{\underline{\tau}^{ca}, \underline{\tau}^{ca} + 1, \cdots, \bar{\tau}^{ca}\}$ *and* $k \in \mathcal{I} \times \mathcal{J}$.

Remark 7.5.1 We observe that the optimization in Theorem 7.5.1 is again non-convex due to the BMI constraints. Similarly, the algorithm in Theorem 7.3.1 can be employed to deal with the bilinearity and obtain an optimal H_2 dynamic output feedback or state feedback controller.

7.6 Illustrative Example

Consider a system (3.2.1)-(3.2.3) with the following parameters

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.1 & 0.4 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, C_1 = [\,0 \quad 1\,],$$

$$D_{12} = [\,0\,], \; C_2 = [\,1 \quad -0.5\,], \; D_{11} = 0.5, \; D_{21} = 0.1, \; D_{22} = 0.$$

It is known that the optimal H_2 control performance for the above system without communication delays is 0.51. Now suppose there exists communication delay from the sensor to the controller and a second order dynamic output feedback controller of the form of (7.3.6) is sought to achieve the optimal H_2 performance. We consider the H_2 control problem for two cases $\bar{\tau}^{sc} = 1$, $\bar{\tau}^{sc} = 2$. When the upper bound of the delay equals 1, the optimal H_2 cost of 0.77 can be obtained by using Theorem

7.3.1 and the controller is given by

$$\Theta = \left[\begin{array}{cc|c} 0.2990 & -0.0060 & -0.0097 \\ 0.2880 & -0.3024 & 0.0922 \\ \hline 0.0059 & 0.0062 & -0.1431 \end{array}\right].$$

When $\bar{\tau}^{sc} = 2$, we arrive at the following switching controllers

$$\Theta_1 = \left[\begin{array}{cc|c} 0.3807 & -0.0147 & -0.0060 \\ 0.3500 & -0.3775 & 0.0291 \\ \hline 0.0502 & 0.0546 & -0.1395 \end{array}\right]$$

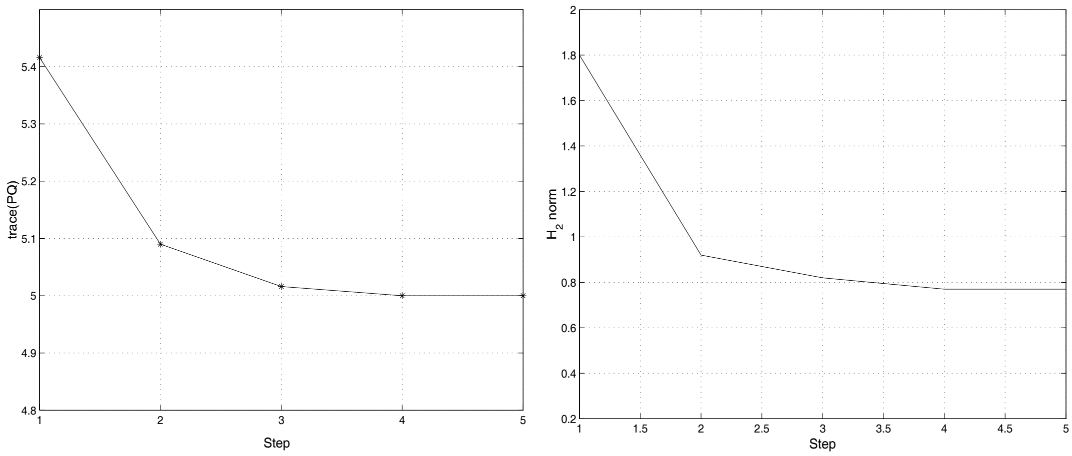
$$\Theta_2 = \left[\begin{array}{cc|c} 0.4081 & -0.0074 & -0.0017 \\ 0.2703 & -0.3279 & 0.0076 \\ \hline 0.0250 & 0.0304 & -0.3672 \end{array}\right]$$

by using Theorem 7.3.1 and the optimal H_2 cost is 1.1 under the above controller.

The convergence of the proposed algorithm is clearly demonstrated in Figures 7.2 and 7.3 for $\bar{\tau}^{sc} = 1$ and $\bar{\tau}^{sc} = 2$ respectively. From Figure 7.2.1, we can see that $trace(P^k Q^k)$ is a strictly decreasing sequence and after 4 steps it converges to $n_x = 5$ which is the dimension of the matrix P. Figure 7.3.1 also shows that the sequences $trace(P_i^k Q_i^k)$, $i = 1, 2$, strictly decrease and both converge to the value of the dimension of the matrix P. It can be seen from Figures 7.2.2 and 7.3.2 that the H_2 norm also decreases for each case of $\bar{\tau}^{sc} = 1$ and $\bar{\tau}^{sc} = 2$.

7.7 Conclusion

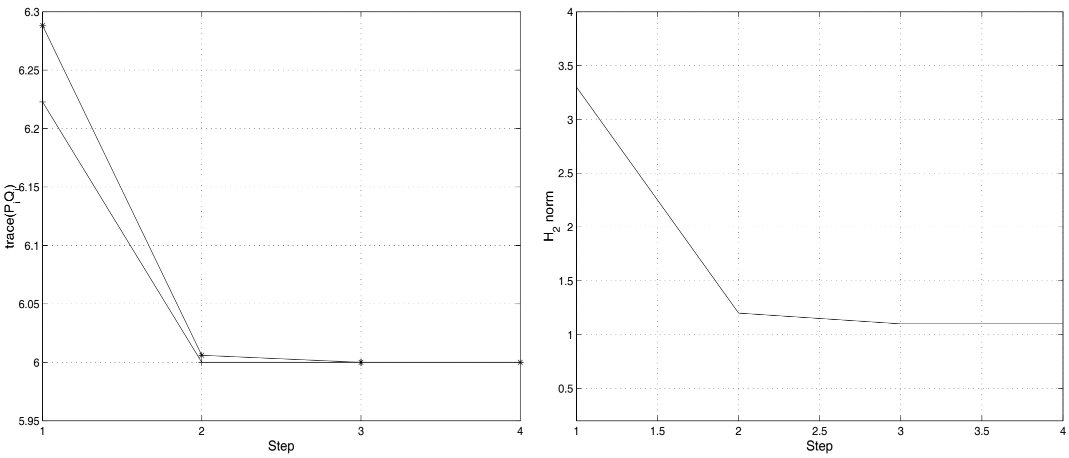
This chapter has presented an iterative LMI algorithm for designing H_2 controllers for discrete-time systems with random but bounded communication delays in the



7.2.1 The convergence of $trace(PQ)$ 7.2.2 The convergence of the H_2 norm

Figure 7.2 The convergence for the case of $\bar{\tau}^{sc} = 1$.

feedback loop. By output and/or input augmentation, the networked control problem with random delay was formulated as a switched static feedback control problem. Then the methods in switched control were employed to derive controllers in terms of bilinear matrix inequalities (BMIs). The sequential linear programming matrix method (SLPMM) was used to solve the BMI problem. The design method allows a fixed order controller.



7.3.1 The convergence of $trace(P_iQ_i)$, $i = 1, 2$

7.3.2 The convergence of the H_2 norm

Figure 7.3 The convergence for the case of $\bar{\tau}^{sc} = 2$.

Chapter 8

Stabilization of Networked Control Systems

8.1 Introduction

In this chapter, we continue our study on networked control systems (NCS) described in Figure 6.1 with communication delays. Our objective here is to present simpler methods for the stabilization of NCS. By augmenting the control input or the output of the plant, we formulate the problem as a static output feedback stabilization problem. Then based upon the switched system theory, stabilizing controllers are derived in terms of LMIs for the cases of static and dynamic output feedback, which is in contrast with the BMI approach of the last chapter.

The system model considered in this chapter is given by

$$x(k+1) = Ax(k) + Bu(k) \quad (8.1.1)$$

$$y(k) = Cx(k) \quad (8.1.2)$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^m$ is the input and $y \in \mathcal{R}^p$ is the output. We assume that the output matrix C is of full row rank, which accounts for the linear independence of the components of the output vector $y(k)$. This assumption does not cause any loss of generality since it can be achieved by discarding the redundant components of the output $y(k)$ [DRI02].

8.2 The case with Delays from Sensor to Controller

In this section we will consider the case where there is a digital link between the sensor and the controller. In this case, there will be random transmission delays from the sensor to the controller. The sensor data are assumed to be time stamped and the time delay is random but bounded. We assume that no delay will occur from the controller to the actuator.

8.2.1 Static Output Feedback

Since the sensor output is time stamped, we introduce a mode dependent feedback control law

$$u(k) = K_{\tau^{sc}} y(k - \tau^{sc}) \tag{8.2.1}$$

where τ^{sc} is the transportation delay of the network from the sensor to the controller which is time-varying but is known to satisfy

$$1 \leq \tau^{sc} \leq \bar{\tau}^{sc} < \infty.$$

For the convenience of expression, we assume the minimum time delay is 1.

Denote $\tau^{sc} = i$ and

$$\rho(k) = K_i y(k) = K_i C x(k). \tag{8.2.2}$$

By introducing the augmented state

$$\xi_i(k) = [x^T(k) \quad \rho^T(k-1) \ \cdots \ \rho^T(k-i) \ \cdots \ \rho^T(k-\bar{\tau}^{sc})]^T$$

from (8.1.1)-(8.1.2) and (8.2.2), we have the closed-loop system

$$\xi_i(k+1) = (\bar{A}_i + \bar{B}K_i\bar{C})\xi_i(k) \tag{8.2.3}$$

where $\bar{A}_i \in \mathcal{R}^{(n+\bar{\tau}^{sc}p)\times(n+\bar{\tau}^{sc}p)}$, $\bar{B} \in \mathcal{R}^{(n+\bar{\tau}^{sc}p)\times m}$, $\bar{C} \in \mathcal{R}^{p\times(n+\bar{\tau}^{sc}p)}$ are given by

$$\bar{A}_i = \begin{bmatrix} A & 0 & \cdots & 0 & B & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8.2.4}$$

$$\bar{C} = [C \quad 0 \cdots 0]. \tag{8.2.5}$$

Since the matrix C is of full row rank, so is the matrix $\bar{C}$. Hence, there exists a linear transformation matrix T such that $\mathcal{C} = \bar{C}T = [I_{p\times p} \quad 0 \ \cdots \ 0]$. In this case, denoting $\xi_i = T\eta_i$, the closed-loop system (8.2.3) can be rewritten as

$$\eta_i(k+1) = (\mathcal{A}_i + \mathcal{B}K_i\mathcal{C})\eta_i(k) \tag{8.2.6}$$

where

$$\mathcal{A}_i = T^{-1}\bar{A}_iT, \quad \mathcal{B} = T^{-1}\bar{B}. \tag{8.2.7}$$

We note that in the case of state feedback, the above transformation $T = I$.

As discussed in the last chapter, here the time delay τ^{sc} also satisfies (7.3.9). Then we denote $\tilde{A}_i = \mathcal{A}_i + \mathcal{B}K_i\mathcal{C}$ and the system (8.2.6) can be viewed as a switched system with the switching rule as

$$\tilde{A}_i \to \tilde{A}_j, \ j = 1, 2, \cdots, i+1. \tag{8.2.8}$$

We have the following stability result on the switched system (8.2.6).

Theorem 8.2.1 *The following statements which guarantee the asymptotic stability of the switched system (8.2.6) are equivalent:*

(a) There exist $\bar{\tau}^{sc}$ positive definite matrices $S_1, \ S_2, \ \cdots, S_{\bar{\tau}^{sc}}$, satisfying

$$\tilde{A}_i S_i \tilde{A}_i^T - S_j < 0, \quad i = 1, 2, \cdots, \bar{\tau}^{sc}; \ j = 1, 2, \cdots, i+1. \tag{8.2.9}$$

(b) There exist $\bar{\tau}^{sc}$ symmetric matrices $S_1, \cdots, S_{\bar{\tau}^{sc}}$ and $\bar{\tau}^{sc}$ matrices $G_1, \cdots, G_{\bar{\tau}^{sc}}$ satisfying

$$\begin{bmatrix} -S_j & \tilde{A}_i G_i \\ G_i^T \tilde{A}_i^T & S_i - (G_i + G_i^T) \end{bmatrix} < 0 \tag{8.2.10}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc}; \ j = 1, 2, \cdots, i+1$.

(c) There exist matrices $S_i, \ F_i$ and G_i, $i = 1, \cdots, \bar{\tau}^{sc}$, where S_i is symmetric, such that

$$\begin{bmatrix} -S_j + \tilde{A}_i F_i + F_i^T \tilde{A}_i^T & -F_i^T + \tilde{A}_i G_i \\ -F_i + G_i^T \tilde{A}_i^T & S_i - (G_i + G_i^T) \end{bmatrix} < 0 \tag{8.2.11}$$

for $i = 0, 1, \cdots, \bar{\tau}^{sc}; \ j = 0, 1, \cdots, i+1$.

Proof The equivalence between (a) and (b) has been established by [DRI02]. We show that (a) is equivalent to (c). In fact, if (a) holds, (c) will be satisfied by setting $F_i = 0$ and $G_i = G_i^T = S_i$ and applying the Schur complement. On the other hand, if (8.2.11) holds for some (F_i, Q_i, G_i), multiplying (8.2.11) from the left and from the right by Υ^T and Υ, respectively, where

$$\Upsilon = \begin{bmatrix} I & 0 \\ \tilde{A}_i^T & I \end{bmatrix},$$

the (1,1) and (2,2) blocks of the resulting matrix inequality are respectively

$$-S_j + \tilde{A}_i(G_i + G_i^T)\tilde{A}_i^T < 0$$

and

$$S_i - (G_i + G_i^T) < 0.$$

The above implies (8.2.9).

The equivalence between (b) and (c) of Theorem 1 implies that they have the same degree of conservatism. However, when applying the result to the static output feedback design, (c) can result in a less conservative design. This is because for a static output feedback, an LMI based solution usually requires an equality constraint. In this case, an LMI result with an equality constraint is only sufficient and with more free parameters in (c) a less conservative design can be achieved.

Theorem 8.2.2 *The NCS is stabilizable via a static output feedback, if for some scalars* λ_i, $i = 1, 2, \cdots, \bar{\tau}^{sc}$ *and given matrices* $\Gamma_i \in \mathcal{R}^{p\times(n+\bar{\tau}^{sc}p)}$ *with full row rank, there exist positive definite matrices* $S_i \in \mathcal{R}^{(n+\bar{\tau}^{sc}p)\times(n+\bar{\tau}^{sc}p)}$ *and matrices* $U_i \in \mathcal{R}^{m\times p}$, F_i *and* G_i *of the forms*

$$F_i = \begin{bmatrix} \lambda_i V_i \Gamma_i \\ F_{i1} \end{bmatrix}, \quad G_i = \begin{bmatrix} V_i \Gamma_i \\ G_{i1} \end{bmatrix} \tag{8.2.12}$$

where $F_{i1},\ G_{i1} \in \mathcal{R}^{(n+(\bar{\tau}^{sc}-1)p)\times(n+\bar{\tau}^{sc}p)}$ *and* $V_i \in \mathcal{R}^{p\times p}$ *such that*

$$\begin{bmatrix} -S_j + \mathcal{A}_i F_i + F_i^T \mathcal{A}_i^T + \lambda_i(\mathcal{B}U_i\mathcal{C} + \mathcal{C}^T U_i^T \mathcal{B}^T) & -F_i^T + \mathcal{A}_i G_i + \mathcal{B}U_i\mathcal{C} \\ (-F_i^T + \mathcal{A}_i G_i + \mathcal{B}U_i\mathcal{C})^T & S_i - (G_i + G_i^T) \end{bmatrix} < 0 \tag{8.2.13}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc};\ j = 1, 2, \cdots, i+1.$

In this situation, a static feedback control gain can be given by

$$K_i = U_i V_i^{-1} \tag{8.2.14}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc}.$

Proof First, note that F_i and G_i of the form (8.2.12) implies

$$\mathcal{C}G_i = V_i\Gamma_i, \quad \mathcal{C}F_i = \lambda_i V_i \Gamma_i. \tag{8.2.15}$$

Substituting the above into (8.2.11) and noting that $\tilde{A}_i = \mathcal{A}_i + \mathcal{B}K_i\mathcal{C}$, (8.2.13) follows by letting $U_i = K_i V_i$. It can be observed from (8.2.13) that G_i is invertible, so is V_i. Hence, (8.2.14) holds.

Remark 8.2.1 Note that for given λ_i and Γ_i, (8.2.13) is an LMI and can be solved by convex optimization. Γ_i can be chosen as $\Gamma_i = \mathcal{C}$. Observe that if Theorem 8.2.1(b) is employed, $F_i = 0$ and $\lambda_i = 0$ in (8.2.13), resulting in a more restrictive optimization. Hence, it would be expected that the result of Theorem 8.2.2 would be less conservative.

8.2.2 Dynamic Output Feedback

We introduce a dynamic output feedback controller with state space realization of the form

$$\hat{x}(k+1) = \hat{A}_{\tau^{sc}}\hat{x}(k) + \hat{B}_{\tau^{sc}}y(k-\tau^{sc}) \quad (8.2.16)$$

$$u(k) = \hat{C}_{\tau^{sc}}\hat{x}(k) + \hat{D}_{\tau^{sc}}y(k-\tau^{sc}) \quad (8.2.17)$$

where $\hat{x}(k) \in \mathcal{R}^{\hat{n}}$ is the state of the controller, $\hat{A}_{\tau^{sc}}$, $\hat{B}_{\tau^{sc}}$, $\hat{C}_{\tau^{sc}}$, $\hat{D}_{\tau^{sc}}$ are the controller matrices to be determined which are dependent on the delay τ^{sc}. As in the last section, we consider that the measurement is time stamped. That is, τ^{sc} is known by the controller when the measurement $y(k-\tau^{sc})$ is received. Here, the controller order is not necessarily equal to the plant order.

Denote $\tau^{sc} = i$, $1 \le \tau^{sc} \le \bar{\tau}^{sc}$. We gather all the controller parameters into the following compact form

$$\Theta_i = \begin{bmatrix} \hat{A}_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix}. \quad (8.2.18)$$

Denoting $\rho(k) = Cx(k)$ and

$$\xi_i(k) = \begin{bmatrix} \hat{x}^T(k) & x^T(k) & \rho^T(k-1) & \cdots & \rho^T(k-i) & \cdots & \rho^T(k-\bar{\tau}^{sc}) \end{bmatrix}^T$$

we have the following closed-loop system:

$$\xi_i(k+1) = \begin{bmatrix} \hat{A}_i & 0 & [& \hat{B}_iW_i & &] \\ B\hat{C}_i & A & [& B\hat{D}_iW_i & &] \\ 0 & C & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} \xi_i(k). \quad (8.2.19)$$

The matrix W_i is defined as follows:

$$W_i = [\,\underbrace{0 \quad \cdots \quad I \quad \cdots \quad 0}_{\bar{\tau}^{sc} \text{ terms}} \quad 0\,] \in \mathcal{R}^{p\times(n+\bar{\tau}^{sc}p)} \tag{8.2.20}$$

with the i-th element being an identity matrix.

We can rewrite the above system into the following form:

$$\xi_i(k+1) \quad = \quad (\bar{A}+\bar{B}\Theta_i\bar{C}_i)\xi_i(k) \tag{8.2.21}$$

where $i = 1, 2, \cdots, \bar{\tau}^{sc}$ and

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & 0 & \cdots & 0 & 0 \\ 0 & C & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} I & 0 \\ 0 & B \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & W_i \end{bmatrix}.$$

From the system (8.2.21), we can see that the dynamic output feedback control problem is again converted to a static output feedback control problem.

Observe from $\bar{C}_i$ that there exists a linear transformation matrix T_i such that

$$\mathcal{C}_i = \bar{C}_i T_i = [I \quad 0 \quad \cdots \quad 0].$$

Hence, denoting $\eta_i = T_i\xi_i$, the system (8.2.21) can be rewritten as

$$\eta_i(k+1) = (\mathcal{A}_i + \mathcal{B}_i\Theta_i\mathcal{C}_i)\eta_i(k) \tag{8.2.22}$$

where

$$\mathcal{A}_i = T_i^{-1}\bar{A}T_i, \quad \mathcal{B}_i = T_i^{-1}\bar{B}.$$

It is clear that the closed-loop system (8.2.22) is of the same form as that of (8.2.6). Hence, the following result can be obtained.

Theorem 8.2.3 *The NCS is stabilizable via dynamic output feedback, if for some scalars* $\lambda_i,\ i = 1, 2, \cdots, \bar{\tau}^{sc}$ *and given matrices* $\Gamma_i \in \mathcal{R}^{(\hat{n}+p)\times(n+\hat{n}+(\bar{\tau}^{sc}-1)p)}$ *with full row rank, there exist symmetric matrices* S_i *and matrices* U_i*,* F_i *and* G_i *of the form*

$$F_i = \begin{bmatrix} \lambda_i V_i \Gamma_i \\ F_{i1} \end{bmatrix}, \quad G_i = \begin{bmatrix} V_i \Gamma_i \\ G_{i1} \end{bmatrix}, \tag{8.2.23}$$

where $F_{i1},\ G_{i1} \in \mathcal{R}^{(n+\bar{\tau}^{sc}p)\times(n+\hat{n}+(\bar{\tau}^{sc}-1)p)}$ *and* $V_i \in \mathcal{R}^{(\hat{n}+p)\times(\hat{n}+p)}$ *such that*

$$\begin{bmatrix} -S_j + \mathcal{A}_i F_i + F_i^T \mathcal{A}_i^T + \lambda_i(\mathcal{B}_i U_i \Gamma_i + \Gamma_i^T U_i^T \mathcal{B}_i^T) & -F_i^T + \mathcal{A}_i G_i + \mathcal{B}_i U_i \Gamma_i \\ (-F_i^T + \mathcal{A}_i G_i + \mathcal{B}_i U_i \Gamma_i)^T & S_i - (G_i + G_i^T) \end{bmatrix} < 0 \tag{8.2.24}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc};\ j = 1, 2, \cdots, i+1$*. In this situation, a suitable dynamic output feedback controller can be given by*

$$\Theta_i = U_i V_i^{-1} \tag{8.2.25}$$

for $i = 1, 2, \cdots, \bar{\tau}^{sc}$*.*

Proof First, note that F_i and G_i of the form (8.2.23) implies

$$\mathcal{C} G_i = V_i \Gamma_i, \quad \mathcal{C} F_i = \lambda_i V_i \Gamma_i. \tag{8.2.26}$$

Substituting (8.2.26) into (8.2.11) and noting that $\tilde{A}_i = \mathcal{A}_i + \mathcal{B}\Theta_i\mathcal{C}$, (8.2.24) follows by letting $U_i = \Theta_i V_i$.

8.3 The case with Delays from Controller to Sensor

In this section, we consider the case in which there only exist delays from the controller to the actuator. Since the information of the time delays cannot be known to the controller, a switching control cannot be applied. So a controller of the following form is employed

$$\begin{aligned}\hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k)\\ u(k) &= \hat{C}\hat{x}(k-\tau^{ca}) + \hat{D}y(k-\tau^{ca})\end{aligned}$$

where $\hat{x}(k) \in \mathcal{R}^{\hat{n}}$ is the state of the controller, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ are the controller matrices to be determined and $\tau^{ca} = i$, $1 \leq \tau^{ca} \leq \bar{\tau}^{ca}$ denotes the delays from the controller to the actuator. Denoting $\eta(k+1) = \hat{C}\hat{x}(k)$ and

$$\zeta(k) = [\,\hat{x}^T(k) \quad x^T(k) \quad \eta^T(k-1) \quad \cdots \quad \eta^T(k-i) \quad \cdots \quad \eta^T(k-\bar{\tau}^{ca})\,]^T,$$

we have the following closed-loop system:

$$\zeta(k+1) = \begin{bmatrix} \hat{A} & \hat{B}C & 0 & \cdots & 0 & 0\\ 0 & A & [& BU_i & &]\\ \hat{C} & \hat{D}C & 0 & \cdots & 0 & 0\\ 0 & 0 & I & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}\zeta(k). \tag{8.3.1}$$

The matrix U_i is defined as follows:

$$U_i = \underbrace{[\,0 \quad \cdots \quad I \quad \cdots \quad 0\,]}_{\bar{\tau}^{ca}\ \text{terms}} \tag{8.3.2}$$

with the i-th element being an identity matrix.

By gathering all the controller parameters into the following compact form

$$\Theta = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}, \tag{8.3.3}$$

we can rewrite the closed-loop system (8.3.1) into the following form

$$\zeta(k+1) \quad = \quad (\tilde{A}_i + \tilde{B}\Theta\tilde{C})\zeta(k) \tag{8.3.4}$$

where $i = 1, 2, \cdots, \bar{\tau}^{ca}$ and

$$\tilde{A}_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & [& BU_i & &] \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & C & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Since the matrix C is of full row rank, so is the matrix $\tilde{C}$. Hence, there exists a linear transformation matrix T such that $\mathcal{C} = \tilde{C}T = [I \quad 0 \quad \cdots \quad 0]$. Then the closed-loop system (8.3.4) can be rewritten as

$$\eta(k+1) = (\mathcal{A}_i + \mathcal{B}\Theta\mathcal{C})\eta(k) \tag{8.3.5}$$

where

$$\mathcal{A}_i = T^{-1}\tilde{A}_iT, \quad \mathcal{B} = T^{-1}\tilde{B}.$$

Theorem 8.3.1 *The NCS is stabilizable via dynamic output feedback, if for some scalars λ_i, $i = 1, 2, \cdots, \bar{\tau}^{ca}$ and given matrices $\Gamma_i \in \mathcal{R}^{(\hat{n}+p)\times(n+\hat{n}+(\bar{\tau}^{ca}-1)p)}$ with full row rank, there exist symmetric matrices S_i and matrices U, F_i and G_i of the form*

$$F_i = \begin{bmatrix} \lambda_i V\Gamma_i \\ F_{i1} \end{bmatrix}, \quad G_i = \begin{bmatrix} V\Gamma_i \\ G_{i1} \end{bmatrix}, \tag{8.3.6}$$

where $F_{i1},\ G_{i1} \in \mathcal{R}^{(n+\bar{\tau}^{ca}p)\times(n+\hat{n}+(\bar{\tau}^{ca}-1)p)}$ *and* $V \in \mathcal{R}^{(\hat{n}+p)\times(\hat{n}+p)}$ *such that*

$$\begin{bmatrix} -S_j + \mathcal{A}_i F_i + F_i^T \mathcal{A}_i^T + \lambda_i(\mathcal{B}_i U \Gamma_i + \Gamma_i^T U^T \mathcal{B}_i^T) & -F_i^T + \mathcal{A}_i G_i + \mathcal{B}_i U \Gamma_i \\ (-F_i^T + \mathcal{A}_i G_i + \mathcal{B}_i U \Gamma_i)^T & S_i - (G_i + G_i^T) \end{bmatrix} < 0 \qquad (8.3.7)$$

for $i = 1, 2, \cdots, \bar{\tau}^{ca};\ j = 1, 2, \cdots, \bar{\tau}^{ca}$.

In this situation, a suitable dynamic output feedback controller can be given by

$$\Theta = UV^{-1} \qquad (8.3.8)$$

For the case of static output feedback, similar results can be obtained easily.

8.4 Illustrative Examples

8.4.1 Static Output Feedback

Consider a linear system in the form of (2.2.7)-(2.2.8) with the following parameters:

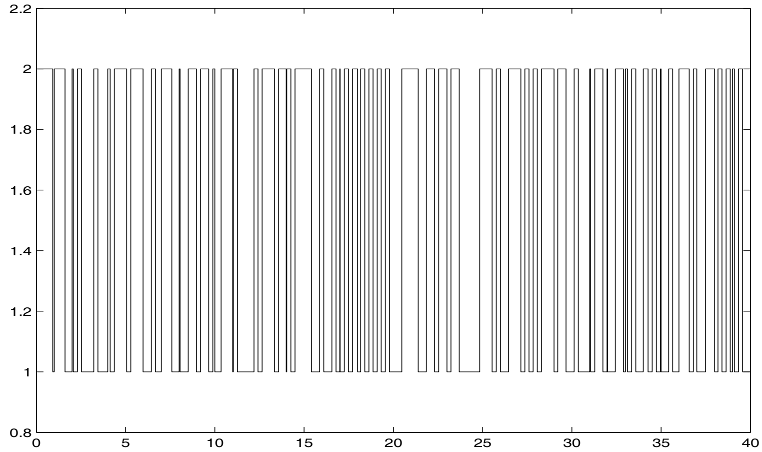
$$A = \begin{bmatrix} 1.1 & 0.1 \\ -0.3 & 0.5 \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 1 \end{bmatrix},\ C = [1 \quad 1].$$

The output of the system is transmitted via a network to the controller. There is a direct link between the controller and the plant. The sensor measurement is time-stamped. So a switching control can be applied. We want to design a static output feedback controller to stabilize the above NCS. When the maximum delay $\bar{\tau}^{sc} = 2$, by choosing $\Gamma_1 = \Gamma_2 = [1\ 0\ 0\ 0]$ we can obtain the switching controller

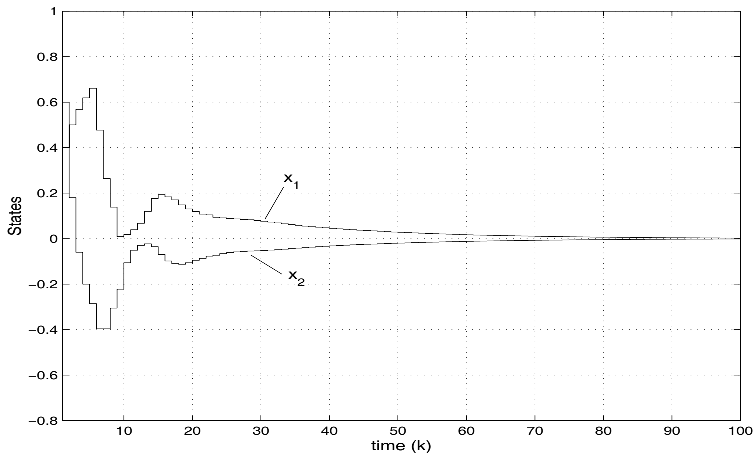
$$K_1 = -0.0568, \quad K_2 = -0.0553.$$

The random time delay and the state trajectories are shown in Figure 8.1, where the initial condition $x_0 = [0.4\ 0.6]$. When the maximum delay $\bar{\tau}^{sc} = 3$, we can obtain the switching controller

$$K_1 = -0.0985, \quad K_2 = -0.0958, \quad K_3 = -0.0984.$$



8.1.1 Random time delay



8.1.2 Initial condition response of the output

Figure 8.1 Random delays and initial condition responses of the states.

8.4.2 Dynamic Output Feedback

Consider another discrete-time system in the form of (2.2.7)-(2.2.8) with the following parameters:

$$A = \begin{bmatrix} 1.0 & 0.1 \\ 0 & -0.5 \end{bmatrix}, \; B = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}, \; C = [\,0.1 \quad 0.2\,].$$

The output of the system is also transmitted via a network to the controller and is time-stamped. We will consider the case of the maximum delay $\bar{\tau}^{sc} = 2$. A dynamic output feedback switching controller will be designed to stabilize the above NCS. First, Γ_1 and Γ_2 in Theorem 8.2.2 are chosen as

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0.01 & 0.01 \\ 0 & 1 & 0 & 0.2 & 0.8 & 0.5 \\ 0 & 0 & 1 & 0 & 0.4 & 0.9 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 1 & 0 & 0 & 1.3 & 0.021 & 0.017 \\ 0 & 1 & 0 & 0.2 & 1 & 0.05 \\ 0 & 0 & 1 & 0 & 0.04 & 0.92 \end{bmatrix}.$$

Then, we can obtain the switching controller parameters

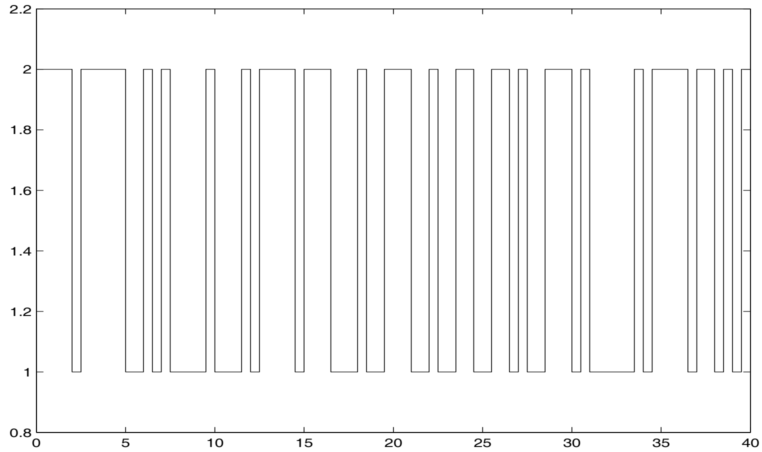
$$\Theta_1 = \left[\begin{array}{cc|c} 0.0905 & -1.5730 & -2.3848 \\ -0.0302 & 0.2614 & 0.5971 \\ \hline 0.7278 & -2.1208 & -3.5556 \end{array}\right]$$

$$\Theta_2 = \left[\begin{array}{cc|c} 0.1030 & 0.6429 & -1.1068 \\ 0.1041 & -0.3703 & -0.0741 \\ \hline 1.0298 & 0.6166 & -2.1705 \end{array}\right]$$

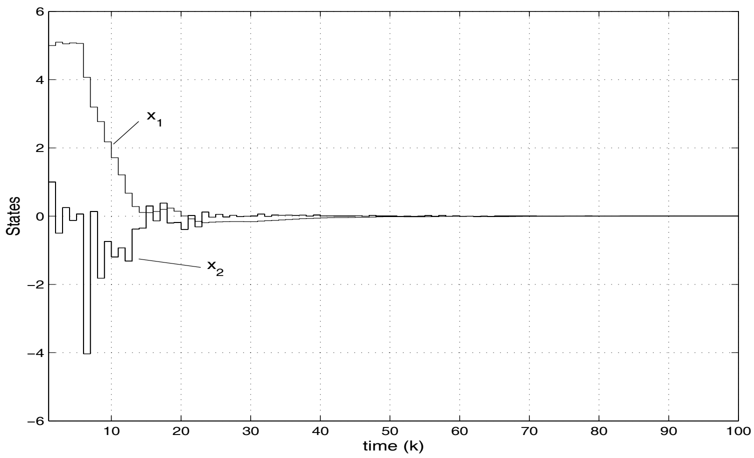
The random time delay and the state trajectories are shown in Figure 8.2, where the initial condition $x_0 = [5 \; 1]$.

8.5 Conclusion

In this chapter, the problem of stabilization of NCS was investigated. Based upon the switched system theory, both the static and dynamic output feedback were considered and LMI based solutions were derived. The proposed LMI approach avoids the equality constraints and iterative approaches which are commonly used in dealing with the stabilization of NCS. The controller parameters can be obtained directly by solving linear matrix inequalities. Two examples are given to illustrate the effectiveness of the method.



8.2.1 Random time delay



8.2.2 Initial condition Response of the output

Figure 8.2 Random delays and initial condition responses of the states.

Chapter 9

Conclusions and Recommendations for Further Research

9.1 Conlusions

In this thesis, we have investigated the control and filtering problems for uncertain discrete-time systems and networked control systems.

In Chapter 2, we have proposed LMI conditions for robust guaranteed cost control with pre-specified disc pole location. The conservatism is reduced due to the use of parameter-dependent Lyapunov functions. An iterative algorithm has also been given to further reduce the conservatism. We have also developed a DLMI method to tackle the GCC problem over the finite horizon for linear time-varying systems.

In Chapter 3, we have solved the problem of dynamic output feedback control of discrete-time systems with polytopic uncertainties which has not been addressed in existing literature. The H_2 and H_∞ performance indices have been used and a unified treatment of designing controllers has been proposed. The parameters of the H_2 or H_∞ controller are given in terms of linear matrix inequalities.

In Chapter 4, we have presented the robust minimum variance filtering for discrete-time linear systems with polytopic uncertainty. Less conservative designs of H_2 and H_∞ filters have been obtained in terms of improved LMIs than existing approaches. We have also proposed an iterative approach to further improve the filter performance.

In Chapter 5, we have discussed the robust H_∞ and H_2 filtering problems for uncertain discrete-time linear systems under limited communication constraints. The limited bandwidth in the network is described by the communication sequence. Then a direct approach to periodic systems is employed to convert the problems to a set of LMIs which can give an explicit expression of the filter for a given sequence. Finally a heuristic search method has been proposed to obtain the desired sequence and the corresponding filter parameters simultaneously.

In Chapter 6, we have developed the optimal H_∞ and H_2 control for networked control systems. Given a sequence, the controller parameters are obtained by solving a set of LMIs. Also a heuristic search method has been given to get the desired sequence and the corresponding controller parameters simultaneously.

In Chapter 7, we have studied the problem of designing H_2 controllers for discrete-time systems with random but bounded communication delays in the feedback loop. By output and/or input augmentation, the problem has been formulated as a switched static feedback control problem. The controller design involves bilinear matrix inequalities (BMIs). Finally the sequential linear programming matrix method (SLPMM) is extended to solve the BMI problem and obtain the parameters of the controller.

In Chapter 8, we have dealt with the problem of stabilization for networked control systems with random but bounded delays. Based on the switched system theory, a simple LMI based solution is presented.

9.2 Recommendations for Further Research

In this section, we recommend some topics for further research.

- In Chapter 3, the output feedback control of polytopic uncertain systems has been studied via an LMI based approach. However the results obtained are somehow primitive. More efficient and less conservative methods deserve further investigation in the future.

- In Chapters 5 and 6, heuristic search methods were proposed to study the optimal filtering and control problems for systems with communication constraints. However, theoretical analysis on the performance and properties of the heuristic algorithm has not been given. In the future, we will study this problem.

- More efficient algorithms to obtain the optimal communication sequence and the corresponding H_∞ and H_2 filters/controlers will be investigated.

- In order to optimize the NCS performance, estimators are used to compensate for network-induced delays and to reduce network traffic in [BW00, YTS02, OMT02]. In the future, we will find new appropriate estimators and strategies to improve the NCS performance.

- In this thesis, we have used discrete-time system models to study the networked control systems (NCS). It is more natural to describe the NCS from the viewpoint of sampled data systems. Issues related to the stability and performance of sampled data systems with communication constraint will be investigated in the future.

Author's Publications

1. Lilei Lu, Lihua Xie and Wenjian Cai, "H_2 controller design for networked control systems," *Asian Journal of Control*, pp. 88-96, Vol 6, No. 1, 2004.

2. Lihua Xie, Lilei Lu, David Zhang and Huanshui Zhang, "Improved robust H_2 and H_∞ filtering for uncertain discrete-time systems," *Automatica*, pp. 873-880, Vol 40, No. 5, 2004.

3. Lilei Lu, Lihua Xie and Zhi Guo, "Robust guaranteed cost control for discrete-time systems with polytopic uncertainties," *in Proc. of the Third International Conference on Control Theory and Applications*, pp. 301-305, Pretoria, South Africa, Dec. 2001.

4. Lilei Lu, Lihua Xie and Wenjian Cai, "H_2 controller design for networked control systems," *in Proc. of the Fourth Asian Control Conference*, pp. 1590-1595, Singapore, Sept. 2002.

5. Lilei Lu and Lihua Xie, "Robust H_∞ filtering for discrete-time systems with limited communication," *in Proc. of the Fourth International Conference on Control and Automation*, pp. 737-741, Montreal, Canada, Jun. 2003.

6. Lilei Lu, Lihua Xie and Minyue Fu, "Optimal control of networked systems with limited communication: a combined heuristic and convex optimization approach," *in Proc. of the 42nd IEEE Conference on Decision and Control*, pp. 1194-1199, Maui, USA, Dec. 2003.

7. Lihua Xie, Lilei Lu, David Zhang and Huanshui Zhang, "Robust filtering for uncertain discrete-time systems: an improved LMI approach," *in Proc. of the 42nd IEEE Conference on Decision and Control*, pp. 906-911, Maui, USA, Dec. 2003.

8. Lilei Lu, Ran Yang and Lihua Xie, " Robust H_2 and H_∞ Control of Discrete-time Systems with Polytopic Uncertainties via Dynamic Output Feedback," accepted by 2005 American Control Conference.

9. Lilei Lu, Lihua Xie and Minyue Fu, "Optimal control of networked systems: a combined heuristic and convex optimization approach," a revised version submitted to *International Journal of Control.*

10. Lilei Lu and Lihua Xie, "Robust H_∞ and H_2 filtering for discrete-time systems with limited communication," a revised version submitted to *IEEE Trans. on Circuits and Systems-I.*

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