SHORTENING AND TURBO DECODING
OF
PRODUCT CODES

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Shortening and Turbo Decoding
Of
Product Codes

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Teachers
Paint their minds
and guide their thoughts
Share their achievements
and advise their fault

Inspire a Love
of knowledge and truth
As you light the path
Which leads our youth

For our future brightens
with each lesson you teach
Each smile you lengthen
Each goal you help reach

For the dawn of each poet
each philosopher and king
Begins with a Teacher
And the wisdom they bring

Kevin William Huff

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Summary

Product codes (PC) have become an important class of error correction codes in recent years. They have been adopted in the IEEE 802.16 standard for wireless metropolitan area networks (WMAN) and received wide industry acceptance as a good forward error correction (FEC) solution. They exhibit near-capacity bit-error-rate (BER) performance and reasonable decoding complexity with soft decoding. Moreover, unlike convolutional-based Turbo codes, product codes suffer from lower error floors.

In communication systems involving block and product codes, code shortening of a code is commonly used to flexibly adjust the code length or code rate to accommodate protocol. In this research, we first look into the BER performance analysis of shortened and shortened-extended product codes. Our achievement is the derivation of useful code weight enumerator expressions based on the widely known properties of the original code (before extension or shortening) and the number of nullified information symbols. They have been verified to be tight for Hamming, BCH and Golay codes. Such weight enumerator expressions lead naturally to useful decoding BER performance bounds and coding loss formulas.

Secondly, we found that the existing code shortening approach specified in the IEEE 802.16a WiMax standard is not optimal in terms of minimum Hamming distance. Hence we set out to find better code shortening scheme. The result is a novel "Ordered Shortening" scheme which produces much improved code distance as well as code weight enumerator distributions. These improved code properties lead to improved BER performance for the ordered-shortened codes, as verified by computer simulations using
Summary

the Pyndiah decoding algorithm. Furthermore, the proposed ordered shortening scheme is also shown to be able to meet the theoretical upper bound values of minimum Hamming distance for some codes.

Finally, we have investigated the turbo decoding performance of product codes in pulsed or partial-band jamming channels, which are of particular interest to military/defense communication systems. We have developed the BER bounds of turbo product codes in pulsed jamming channel with Rayleigh fading, and optimised the Pyndiah-Chase algorithm for such channels. The results show that with the use of jamming state information and channel state information, the simulated BERs of the product codes are able to approach the asymptotic BER bounds.
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Conventions of Symbols Notations

- Small capital letter and italic, e.g. $x$, to represent a scalar symbol or variable.
- Bold small capital letter and italic e.g. $\mathbf{x}$, to represent a vector.
- Bold big capital letter enclosed with square brackets, e.g. $[X]$, to represent a matrix.
- Big capital letter enclosed with curly brackets and italic, e.g. $\{x\}$, to represent a set.
Product codes have become an important class of error correction codes in recent years. They have been shown to exhibit good bit-error-rate (BER) performance with reasonable decoding complexity. In particular, using a modified-Chase algorithm, Pyndiah [1][2][3] has shown that more than 98% of channel capacity can be achieved with high-code-rate product codes using only 4 iterations on the Gaussian channel and that the BER slope for the Rayleigh channel without using channel state information is as steep as the Gaussian channel. Moreover, unlike convolutional-based Turbo codes, product codes suffer from lower error floors. They have been adopted in the IEEE 802.16 standard for wireless metropolitan area networks (WMAN).

The modified chase algorithm by Pyndiah [1], however has several limitations. Firstly, it uses two heuristic parameters, $\alpha$ and $\beta$ which require optimization through repeated simulations. Secondly, the extrinsic information in the algorithm has been normalized to a mean absolute value of one. Hence, it does not allow for varying channel conditions. However, with both sequences $\alpha$ and $\beta$ made adaptive by [4], the algorithm is applicable to various codes, modulation schemes and channel conditions. Also, [5] has
developed an alternative approach to estimating $\beta$ using the distance properties of the component codes.

Another limitation of modified Chase algorithm by [1] is that the decoding complexity becomes very large when the minimum Hamming distance of the component codes for the product code is large. Hence, with the aim of reducing the complexity, research [6][7] has been done on list-based soft-in-soft-out (SISO) decoding algorithms, which calculate extrinsic information using a list of codewords. The "box and match" decoding algorithm by [8] is also able to reduce the computational complexity of the ordered statistic decoding algorithm at the expense of increased memory requirements.

In digital communication systems with FEC (forward error correction) involving product codes, code extension and shortening, or a combination of both, are commonly used [9]. A code can be shortened by setting certain information bits to zero during encoding and not transmitting these nullified information bits [10]. An extended code can be constructed by appending a single parity check bit on all the bits in the code word [10]. Shortening of a code is usually used to flexibly adjust the code length and code rate, while code extension is used to improve the BER performance with only a slight decrease in the code rate. Similar to code shortening, puncturing is also usually used to adjust the code length and code rate. However, unlike code shortening, code puncturing does not nullify the information symbols and instead stops the transmission of some parity symbols, causing an increase in the code rate.

The BER performance of a FEC code relies critically on the weight enumerator spectrum of the code, hence much research has been conducted on finding the weight enumerators of good codes such as Hamming [12], BCH [12][13][13] and product
codes[11]. Although the full weight spectrum of a Hamming code or BCH code can be found, the full weight spectrum of product code is not known. [14] attempts to give a better estimate of the full weight spectrum of product code. Specifically, it first computes the known (low) weight codewords based on known parameters of the component codes making up the product code; then it estimates the distribution of unknown weight codewords using a binomial approximation. The drawback of the resultant estimated weight spectrum, however, is that the predicted BER at low $E_b/N_0$ tends to be overestimated.

The weight enumerator research mentioned above pertains to un-extended and un-shortened codes. The effect of code extension on the code weight enumerator can be easily deduced [15], but the same cannot be said of code shortening. In [17], the undetected error probability of shortened Hamming codes is developed, and the weight distributions of shortened Hamming code are found through its dual codes. However, the result in [17] is limited to the class of shortened Hamming code and not other shortened codes such as BCH codes and product codes. As product codes in shortened forms have been considered in standards IEEE802.16a, it is important to know analytically how their BER performance will be affected by code shortening. To our knowledge, no such analytical performance studies have been reported in the literature. Hence, we would like to develop useful weight enumerator expression of shortened code obtained from generic block and product codes.

The conventional shortening scheme performs shortening of a code by nullifying a block of contiguous information symbol positions during encoding and not transmitting these nullified information symbols [10]. We have observed that such shortening
Chapter 1 – Introduction

approach is not optimal in terms of the resultant code distance and correspondingly the
decoding bit error probability performance. Hence we would like to find new ways of
code shortening that can provide better code properties and BER performance.

Regarding the practical decoding algorithm for product code, work has been reported
code in [17][18][19] on using the Pyndiah’s decoding algorithm in Gaussian and
Rayleigh channels. In jamming channels, which are of interest to military
communications [21], turbo codes and serially concatenated convolutional codes have
been investigated [20][21], not product codes. We are hence interested to study the
performance of product codes in pulsed jamming channels, in terms of analytical BER
bounds as well as the optimality of practical decoding algorithm based on Pyndiah’s
algorithm. Pulsed jamming channel is a class of jamming channel, which is fairly
“powerful” and hence is of interest [21][34].

1.2 Objectives

Following the motivations mentioned earlier, the 3 main objectives of this thesis are:

a) To derive useful weight enumerator expression for shortened codes and thereafter
apply it to obtain the BER bounds for shortened codes.

b) To develop a new method of code shortening such that the resultant code
properties (minimum Hamming distance and minimum weight enumerator) can
be improved in order to produce better BER performance.

c) To derive the BER bounds of product codes in pulsed jamming channels, and to
extend and optimize Pyndiah’s algorithm for such channels.
1.3 Major Contributions of the Thesis

The major contributions of the thesis include:

a) Closed-form weight enumerator expression has been for conventional shortened and shortened-extended block/product codes. It is expressed in known parameters of its original (un-shortened) codes and hence can be easily obtained. It has been verified to be tight for Hamming, BCH and Golay codes. For small codes, the tightness of the proposed weight enumerator is verified with the true weight enumerator values, while for larger codes, the tightness is verified by deriving the associated BER bounds and comparing with Monte Carlo simulation results. The asymptotic coding loss expressions due to code shortening have also been derived.

b) An improved method of code shortening called Ordered Shortening has been developed for systematic codes. Under this new scheme, we propose to shorten a code by nullifying carefully selected information symbol positions which correspond to codewords with the smallest Hamming distances. As such, the shortened codes from this proposed scheme produce larger $d_{\text{min}}$ (minimum Hamming distance) and/or smaller $A_d$ (code weight enumerator) values than a conventional-shortened code with the same code length and code rate. These improved $d_{\text{min}}$ and $A_d$ properties lead to improved BER performance as verified by computer simulations. Furthermore, the $d_{\text{min}}$ values of the proposed ordered shortening scheme are shown to meet the theoretical upper bound values for some shortened BCH codes.
c) We have derived the BER bound for pulsed jamming channel with Rayleigh fading. We have also extended and optimized Pyndiah’s decoding algorithm to pulsed jamming channels with AWGN and Rayleigh fading. We show that with the proper use of jamming state information and channel state information to obtain the test positions and to compute the extrinsic information, decoding product codes using Pyndiah’s algorithm is able to approach the asymptotic BER bound well. We also show that the \( \alpha \) weighting factors in Pyndiah’s decoding algorithm for large product codes need to be re-optimized in order to produce the best BER results.

1.4 Organization of the Thesis

The thesis is organized in the following way:

Chapter 2 gives an overview of some background theory of linear block codes, product codes, the BER union bounds and the Pyndiah’s soft-in-soft-out (SISO) decoding algorithm for product codes.

Chapter 3 derives the weight enumerator of shortened codes. It then makes the use of the weight enumerator to obtain BER bounds of shortened and shortened-extended codes of linear block codes and product codes. The new bounds developed are verified using simulated results. Coding loss expressions are derived.

Chapter 4 introduces a novel code shortening scheme, called “Ordered Shortening”, and elaborates on its various strength over conventional shortening scheme.
Chapter 5 extends and optimizes the Pyndiah’s decoding algorithm to pulsed jamming channels. The BER bounds for such pulsed jamming channels are also derived. Simulations and analytical BER results are compared and discussed.

Chapter 6 concludes the thesis and provides suggestions for future work.
Chapter 2

Background

2.1 Linear Block Codes

2.1.1 Linear Block Codes

A block code with length \( n \) and \( 2^k \) codewords is called a linear \((n, k)\) code if and only if its \( 2^k \) code words form a \( k \)-dimensional subspace of the vector space of all the \( n \)-tuples over the field \( \text{GF}(2) \) [12]. The encoder of a block code breaks the information sequence into message blocks of \( k \) information symbols each. After encoding, each message block is transformed independently into a codeword of \( n \) symbols. The set of \( 2^k \) codewords of length \( n \) is called a \((n, k)\) code. A \((n, k)\) linear block code has code length \( n \), message length \( k \) and satisfies the following conditions [23]:

1) component-wise modulo-2 sum of two codewords is another codeword and

2) the code contains the all-zero codeword.

The code rate \( r_c \) is given by \( k/n \).

2.1.2 Matrix Description of Linear Block Codes

Note that the addition and multiplication operations involved in Sections 2.1.2 and 2.1.3 refer to the modulo-2 addition and multiplication.
A \((n, k)\) linear block code can be specified by a set of \(k\) linearly independent binary \(n\)-tuples \(g_1, g_2, ..., g_k\). The codewords are obtained as linear combinations of these \(k\) \(n\)-tuples. The \(g_1, g_2, ..., g_k\) vectors can be arranged as rows of a \(k \times n\) matrix \([G]\) called the generator matrix of the code as follows [23]:

\[
[G] = \begin{bmatrix}
    g_1^T \\
    g_2^T \\
    \vdots \\
    g_k^T
\end{bmatrix} = \begin{bmatrix}
    g_{11} & g_{12} & \cdots & g_{1n} \\
    g_{21} & g_{22} & \cdots & g_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    g_{k1} & g_{k2} & \cdots & g_{kn}
\end{bmatrix}.
\]

The codeword \(e = [e_1, e_2, ..., e_n]\) for an information message \(m\) can then be written as

\[
e = m. [G] = m_1 g_1 + m_2 g_2 + \cdots + m_k g_k.
\]

Fig. 2.1. Structure of Systematic Codeword

A systematic code has the additional structure that the message sequence is itself part of the codeword. The \(k\) symbols of the \(n\)-tuple codeword are identical to the information sequence and the remaining \((n-k)\) symbols constitute the parity symbols, as shown in Fig. 2.1. The generator matrix for a systematic block code has the following form

\[
[G] = [P \  I_k] \quad \text{where} \quad [P] \quad \text{is a} \quad k \times (n-k) \quad \text{matrix and} \quad [I_k] \quad \text{is the} \quad k \times k \quad \text{identity matrix. The parity check matrix} \ [H] \quad \text{of the systematic code is then given by} \ [H] = [I_{n-k} \ P^T] \quad \text{where} \quad [P^T] \quad \text{is an} \quad (n-k) \times k \quad \text{matrix given by the transpose of} \ [P].
\]
2.1.3 Minimum Hamming Distance

The Hamming weight of a codeword is equal to the number of non-zero symbols in
the codeword. For linear block codes, the minimum Hamming distance \( d \) of the code is
given by the smallest Hamming weight of the non-zero codewords. Given the parity
check matrix \([H] = [h_1, h_2, \ldots, h_l, \ldots, h_n]\) where \( h_l \) represents the \( l \)th column of \([H]\), a
linear block code is defined by the set of codewords \( \{e | e[H]^T = 0\} \) where 0 refers to a
zero vector [23]. That is, a codeword \( e \) satisfies \( e_1 h_1 + e_2 h_2 + \ldots + e_l h_l + \ldots + e_n h_n = 0 \)
where \( e_l \) is the \( l \)th symbol of the codeword \( e \). As such, the non-zero symbol positions
of the codeword \( e \) correspond to the columns of \([H]\) that sum to 0. Hence, the minimum
Hamming distance of a linear block code is also given by the smallest number of columns
of the matrix \([H]\) that sum to 0 [23]. This minimum Hamming distance parameter of a
block code is important as it determines the error correcting and error detecting capability
of a code.

Note that a linear block code can also be represented by the following convention: \((n, k, d_{\text{min}})\) where \( d_{\text{min}} \) is the minimum Hamming distance of the code.

2.1.4 Weight Enumerator

For a block code having a Hamming distance distribution \( \{d_{\text{min}}, \ldots, d_i, \ldots, d_{\text{max}}\} \), where
the set \( \{d_{\text{min}}, \ldots, d_i, \ldots, d_{\text{max}}\} \) denotes all possible Hamming distances that the code has, it
has a correspondingly weight spectrum \( \{A_{d_{\text{min}}}, \ldots, A_{d_i}, \ldots, A_{d_{\text{max}}}\} \) where \( A_{d_i} \) is the number
of all codewords with Hamming weight \( d_i \). The weight spectrum of a block code is
useful as it is used in obtaining its BER bound for a code.
Chapter 2 – Background

It has been observed that the binomial distribution is a close approximation to the weight distributions of most binary \((n, k)\) linear codes [13]. In [24] and [25], the following binomial approximation is used:

\[
A_d = \begin{cases} 
1, & d = 0, d = n \\
\lambda \binom{n}{d}, & d_{\text{min}} \leq d \leq n - d_{\text{min}} \\
0, & \text{elsewhere} 
\end{cases}
\]  

(2.1)

This distribution is palindromic with \(A_d = A_{n-d}\). \(\lambda\) is chosen such that the total number of codewords of a code is satisfied i.e. \(\sum_d A_d = 2^n\). Therefore \(\lambda\) is given by

\[
\lambda = \frac{2^n - 2}{\sum_{d=d_{\text{min}}}^{n-d_{\text{max}}} \binom{n}{d}}.
\]  

(2.2)

Example

The weight distribution for a Hamming \((7, 4)\) is given by \(A_0 = 1, A_3 = 7, A_4 = 7\) and \(A_7 = 1\) with \(\lambda = 0.2\).

2.1.5 Families of Linear Block Codes Considered in this Thesis

In this thesis, we shall consider the families of Hamming, BCH and their product codes in the context of code shortening and Turbo decoding in jamming channels. Their essential features are summarized below.

2.1.5.1 Hamming Codes

Hamming codes are characterized by the following parameters:
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2.1.5.2 BCH Codes

BCH codes are characterized by the following parameters:

\[
\begin{align*}
\text{Block length:} & \quad n = 2^m - 1 \\
\text{No of message symbols:} & \quad k = 2^m - 1 - m \\
\text{No of parity symbols:} & \quad n-k = m \\
\text{Minimum Hamming distance:} & \quad d_{\text{min}} = 3
\end{align*}
\]  

(2.3)

where \( m \geq 3 \). Hamming codes are single-error correcting binary perfect \(^1\) codes.

2.2 Product Codes

2.2.1 Product Codes

Product codes are serially concatenated codes built up using two or more short block codes. Fig. 2.2 shows the encoder of a product code \( C^p \) made up of two systematic linear block codes \( C( n^h, k^h, d_{\text{min}}^h) \) and \( C( n^v, k^v, d_{\text{min}}^v) \) called the horizontal and vertical component codes respectively.

---

\(^1\) A perfect code is a code which satisfies the Hamming bound \( 2^{n-k} \geq \sum_{i=0}^{t}(\binom{n}{i}) \) with equality, where \( t \) is the error correcting capability of the code.
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The product code \( C( n^p, k^p, d^p_{\text{min}}) = C^h \otimes C^v \) is obtained by [1]:

1) placing \((k^v \times k^h)\) information bits in an array of \(k^v\) rows and \(k^h\) columns;

2) encoding the \(k^v\) rows using code \(C^h\) row by row;

3) encoding the \(n^h\) columns using code \(C^v\) column by column.

Fig. 2.3 shows the resultant 2-dimensional product code. It can be seen that all rows of the product code are codewords of \(C^h\) and all columns of the product code are codewords of \(C^v\).
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The parameters of product code $C^p$ are [1]:

1) code length: $n^p = n^r \times n^h$,

2) message length: $k^p = k^r \times k^h$ and

3) minimum Hamming distance: $d^p_{\text{min}} = d^r_{\text{min}} \times d^h_{\text{min}}$.

Hence product code is capable of building very long block codes with large minimum Hamming distance $d^p_{\text{min}}$ using relatively short codes.

2.2.2 Weight Enumerator of Product codes

Unlike block codes, as the dimensions of product codes are higher, the full distance spectrum and weight enumerator are not easily known. However, for the special case where both component codes are binary Hamming codes of length $n^h$ and $n^r$, the following results have been found [11]:

For $d \leq 17$, the number $A_d$ of weight $d$ codewords in the product of binary Hamming codes of length $n^h$ and length $n^r$ is given by the following formulas:

$$A_d = \begin{cases} 
1, & \text{if } d = 0 \\
A_1(n^h)A_3(n^r), & \text{if } d = 9 \\
A_3(n^h)A_4(n^r) + A_4(n^h)A_3(n^r), & \text{if } d = 12 \\
A_3(n^h)A_4(n^r) + A_4(n^h)A_3(n^r), & \text{if } d = 15 \\
19A_4(n^h)A_4(n^r), & \text{if } d = 16 \\
36A_4(n^h)A_4(n^r), & \text{if } d = 17 \\
0, & \text{otherwise}
\end{cases}$$ (2.5)
where

\[
A_3(n) = \frac{n(n-1)}{6} \\
A_4(n) = \frac{n(n-1)(n-3)}{24} \\
A_5(n) = \frac{n(n-1)(n-3)(n-7)}{120}
\] (2.6)

2.3 Code Shortening/Extension

2.3.1 Extended Code

Given a linear binary \((n, k)\) code, an extended \((n+1, k)\) code can be constructed by appending a single parity check bit on all the bits in the code word [10]. Correspondingly, the code rate is reduced to \(k/(n+1)\). With code extension, all the odd Hamming distance becomes the next even Hamming distance i.e. \(d+1\). Therefore the typically odd minimum Hamming distances of Hamming and BCH codes become even. Also the weight enumerator of the extended code is now given by the summation of \(A_d\) with odd \(d\) and \(A_d\) with the next even \(d\) (i.e. \(d+1\)) i.e. \(A_d + A_{d+1}\). Take the extended-Hamming (e-Hamming) \((8, 4)\) as an example, Tables 2.1 and 2.2 shows the Hamming distance spectrum and weight enumerator of Hamming\((7, 4)\) before and after extension respectively.
Table 2.1. Code Distance and weight distributions of Hamming(7, 4)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$A_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.2. Code Distance and weight distributions of e-Hamming(8, 4)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$A_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

2.3.2 Shortened Code

A linear $(n, k)$ code consisting of $k$ information bits and $(n-k)$ check bits can be shortened into a $(n-s, k-s)$ linear code by setting the first $s$ information bits to zero during encoding and not transmitting these nullified information bits [10]. Hence the length of the transmitted codeword is shortened to $(n-s)$. Correspondingly, the code rate is reduced to $(k-s)/(n-s)$. Also, the minimum Hamming distance of the shortened code is at least as large as the minimum distance of the original un-shortened code [10].


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2.4 Decoding BER Union Bounds

2.4.1 Decoding BER Union Bounds in AWGN Channel

Given a \( (n, k) \) binary linear code with weight enumerator \( A_d \), code-rate \( r_c = k/n \), signal energy per bit \( E_b \) and single-sided power spectral density \( N_0 \), the soft decoding bit error probability \( P_b \) of this code in an Additive-White-Gaussian-Noise (AWGN) channel, assuming coherent binary/quadrature phase shift keying (B/QPSK) modulation, is upper-bounded by \[ P_b \leq \sum_{d} \frac{d}{n} A_d P_d \left( d, r_c \right) \] (2.7)

where

\[ P_d \left( d, r_c \right) = Q \left( \sqrt{2 d r_c E_b N_0} \right) \] (2.8)

\( P_d(d, r_c) \) is called the pair-wise error probability \[23\]. It is the probability that the decoder makes a wrong decision by selecting another codeword, instead of the actual transmitted codeword, as the decoded sequence.

2.4.2 Decoding BER Union Bounds in Rayleigh Fading Channel

When channel state information is available to the channel decoder, the average bit error probability of the code in memoryless Rayleigh fading channel is upper-bounded by

\[ \overline{P}_b \leq \sum_{d} \frac{d}{n} A_d \overline{P}_d \left( d, r_c \right) \] (2.9)

where \( \overline{P}_d \left( d, r_c \right) \) is the average pairwise error probability.
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$P_d(d, r_c)$ can be found by conditioning on the fading coefficient vector $a = [a_1, \ldots, a_j, \ldots, a_n]$ where $a_j$ is the fading amplitude modeled with a Rayleigh probability density function, and then averaging this pairwise error probability over the pdf $p_a(a)$ of $a$. Assuming that the fading coefficients are independent, the conditioned pairwise error probability is given by [26]

$$P_d(d, r_c|a) = Q\left(\sqrt{\frac{2r_cE_b}{N_0\sum_{i=1}^n a_i^2}}\right). \tag{2.10}$$

Therefore the average pairwise error probability is

$$\bar{P}_d(d, r_c) = \int p_a(a)P_d(d, r_c|a)da \tag{2.11}$$

Since $a$ is a multi-dimensional variable,

$$\bar{P}_d(d, r_c) = \int \cdots \int p_a(a_1, a_2, \ldots, a_d)P_d(d, r_c|a_1, a_2, \ldots, a_d) da_1, \ldots, da_d. \tag{2.12}$$

For memoryless fading channel, each $a_i$ is independent, hence

$$p_a(a) = p_a(a_1, a_2, \ldots, a_d) = \prod_{i=1}^d p(a_i) \tag{2.13}$$

where

$$p(a_i) = 2a_ie^{-a_i^2} \tag{2.14}$$

is the probability density function (pdf) for the Rayleigh distribution.

It is noted that $Q(x)$ can be expressed in an alternative form [26] given by

$$Q(x) = \frac{1}{\pi} \int_0^\infty e^{-\frac{x^2}{2\sin^2\theta}} d\phi. \tag{2.15}$$
Using (2.13), (2.14) and (2.16) in (2.12), it can be shown that the $d$-dimensional integral for $P_d(d, r_c)$ reduces to a product of integrals over each $a_t$[26] to give:

$$P_d(d, r_c) = \frac{1}{\pi} \int_0^{\pi/2} \left[ \frac{\sin^2 \phi}{r_c + \sin^2 \phi} \right]^d d\phi$$  \hspace{1cm} (2.16)

Hence, with the following parameters of a code: weight distribution $A_d$, $n$ and $k$, the BER performance bound of a linear block code can be computed via (2.7) or (2.9).

### 2.5 Pyndiah’s SISO Decoding Algorithm of Product Codes for AWGN Channel

Pyndiah [1] has proposed a soft-in-soft-out (SISO) iterative decoding algorithm for product codes and shown that it is able to achieve optimum BER performance for high-rate component codes (with low minimum Hamming distance) such as Hamming codes by sequentially decoding the rows and columns of the product code with feedback. Other list-based decodings have also been developed by [7][Soft-Input Soft-Output List-Based Decoding Algorithm] [8][On Soft-Input Soft-Output Decoding Using “Box and Match” Techniques] and shown to be capable of achieving good performance and complexity tradeoff when lower rate component codes (with high minimum Hamming distance) are used.
In this section, we shall review the essential steps and underlying principles of Pyndiah’s algorithm in the next sections. Because each row or column of the product code is a linear block code (see Section 2.2), we shall consider the soft decoding algorithm of linear block code first before introducing the soft decoding algorithm of the product code.

### 2.5.1 Soft Decoding of Linear Block Code

In the soft decoding of the linear block code, we review the following decoding schemes [1]:

- Maximum-Likelihood decoding (MLD)
- Chase Decoding

#### 2.5.1.1 AWGN Channel Model

Fig. 2.4 shows the model for an AWGN channel. Consider the transmission of binary phase shift keying (BPSK) modulated vector \( \mathbf{x} = (x_1, \ldots, x_m, \ldots, x_n) \) of the codeword \( \mathbf{e} = (e_1, \ldots, e_m, \ldots, e_n) \) encoded by a linear block code \( C(n, k, d_{\text{min}}) \) on an Additive White Gaussian Noise (AWGN) channel. The mappings of symbols 0 → -1 and 1 → +1 are assumed for \( \mathbf{e} \rightarrow \mathbf{x} \).
At the receiver, the received signal $\mathbf{r} = (r_1, \ldots, r_j, \ldots, r_n)$ is given by

$$\mathbf{r} = \mathbf{x} + \mathbf{n}$$

(2.17)

where $\mathbf{n} = (n_1, \ldots, n_j, \ldots, n_n)$ is a noise vector with each noise sample having mean zero and variance $\sigma_n^2$. The pdf of each AWGN noise sample $n_j$ is

$$p(n_j) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp \left[ -\frac{1}{2} \left( \frac{n_j}{\sigma_n} \right)^2 \right].$$

(2.18)

### 2.5.1.2 Maximum Likelihood Decoding

Using the log-likelihood ratio (LLR) concept, the reliability vector generated from $\mathbf{r} = (r_1, \ldots, r_j, \ldots, r_n)$ is given by

$$\Lambda(r_j) = \ln \left( \frac{\Pr \{ x_j = +1 | r_j \}}{\Pr \{ x_j = -1 | r_j \}} \right) = \frac{2}{\sigma_n^2} r_j,$$

(2.19)

Note that if a stationary channel is considered, the reliability of $r_j$ can be simplified to $|r_j|$ as the factor $2 / \sigma_n^2$ is constant for all received symbols.

The hard detector's output, $y$, of $\mathbf{r}$ is then given by:
Using MLD, the optimum decision \( d = (d_1, \cdots, d_n) \) corresponding to \( e \) is given by

\[
y_j = \begin{cases} 0 & \text{if} \ \Lambda(r_j) < 0 \\ 1 & \text{if} \ \Lambda(r_j) \geq 0 \end{cases}
\]

(2.20)

where \( \Lambda(r_j) \) is the decision variable for the \( j \)th symbol. The optimum decision \( d \) is given by the codeword with the minimum squared Euclidean distance between itself and \( r \). As can be seen, the computing complexity of ML decoding increases exponentially with \( k \) when an exhaustive search is used for \( d \). Hence, the exhaustive search is not a realistic solution for large \( k \). Therefore, the sub-optimal Chase algorithm for reducing the number of reviewed codewords is used [1].

### 2.5.1.3 Chase's Selection of Codewords

Chase observed that at high signal-to-noise ratio (SNR), the probability that the ML codeword \( d \) is located in the sphere of radius \( (d_{\min} - 1) \) centered on the hard demodulator output \( y = (y_1, \cdots, y_j, \cdots, y_n) \) given by (2.20) is very high. Hence, the number of reviewed codewords can be limited to those in the sphere of radius \( (d_{\min} - 1) \) centered on \( y \). This set of codewords to be reviewed can be selected using the channel information \( |\Lambda(r_j)| \).
The procedure to identify the set of Chase’s codewords to be reviewed is as follows:

Step 1: Determine \( m = \left\lfloor \frac{d_{\text{min}}}{2} \right\rfloor \) least reliable binary elements of \( y \) with the smallest reliability value of \( \Lambda(r_j) \).

Step 2: Form a set of test patterns \( \{t^r\} \) by permuting the \( 2^m \) possible error patterns identified in step 1.

Step 3: Form the set of test sequences \( \{z^r = y \oplus t^r\} \).

Step 4: Perform hard decoding on \( \{z^r\} \) to obtain the set of candidate codewords \( \{c^r\} \).

Therefore, it can be seen that the set of codewords using the exhaustive search has been limited to the set of \( 2^m \) candidate codewords. Hence the optimum decoding decision \( d \) can be chosen from \( \{c^r\} \) using (2.21).

### 2.5.2 Soft Decoding Output of Linear Block Codes

#### 2.5.2.1 Soft Output - Reliability of Decision \( d \)

After determining the decoding decision \( d \) of a block code (either a row or a column), the reliability of each component of vector \( d \) is computed in order to generate soft decisions at the output of the decoder. The reliability of decision \( d_{ij} \) at the output of the decoder is defined using the log-likelihood ratio (LLR)
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\[ \Lambda(d_j) = \ln \left( \frac{\Pr\{x_j = +1|\mathbf{r}\}}{\Pr\{x_j = -1|\mathbf{r}\}} \right) \]  

(2.23)

Because \( d \) is one of the \( 2^k \) codewords of \( \mathbf{C} \), the numerator of (2.23) can be written as following by considering different codewords of \( \mathbf{C} \)

\[ \Pr\{x_j = +1|\mathbf{r}\} = \sum_{c' \in \{s_j^+\}} \Pr\{x = c'|\mathbf{r}\} \]  

(2.24)

where \( \{s_j^+\} \) is the set of codewords \( \{c'\} \) such that \( c'_j = +1 \) while the denominator of (2.23) can similarly be written in the form

\[ \Pr\{x_j = -1|\mathbf{r}\} = \sum_{c' \in \{s_j^-\}} \Pr\{x = c'|\mathbf{r}\} \]  

(2.25)

where \( \{s_j^-\} \) is the set of codewords \( \{c'\} \) such that \( c'_j = -1 \).

By applying Bayes’ rule to (2.24) and (2.25), and by assuming that the different codewords are uniformly distributed, equation (2.23) can be written as

\[ \Lambda(d_j) = \ln \left( \frac{\sum_{c' \in \{s_j^+\}} p(\mathbf{r}|x = c')} {\sum_{c' \in \{s_j^-\}} p(\mathbf{r}|x = c')} \right) \]  

(2.26)

where

\[ p(\mathbf{r}|x = c') = \left( \frac{1}{\sqrt{2\pi}\sigma_n} \right)^n \exp \left( -\frac{|\mathbf{r} - c'|^2}{2\sigma_n^2} \right) \]  

(2.27)

Combining equation (2.26) and (2.27), the reliability of decision \( d_j \) will be
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\[ \Lambda(d_j) = \ln \left( \frac{\sum_{c' \in \{s_j^i\}} \exp \left( -\frac{|r - c'|^2}{2\sigma_n^2} \right)}{\sum_{c' \in \{s_j^i\}} \exp \left( -\frac{|r - c'|^2}{2\sigma_n^2} \right)} \right) \]  

(2.28)

To simplify the computation, we only consider the two codewords \( c^{+1(j)} \) and \( c^{-1(j)} \), in \( \{s_j^i\} \) and \( \{s_j^i\} \) respectively, which have minimum Euclidean distance from \( r \).

Hence, (2.28) becomes

\[ \Lambda(d_j) = \frac{1}{2\sigma_n^2} \left( |r - c^{+1(j)}|^2 - |r - c^{-1(j)}|^2 \right) + \ln \left( \frac{\sum_i A_i}{\sum_i B_i} \right) \]  

(2.29)

where

\[ A_i = \exp \left( -\frac{|r - c^{+1(j)}|^2 - |r - c'j|^2}{2\sigma_n^2} \right) \text{ with } c' \in \{s^{-1(j)}\} \]  

(2.30)

and

\[ B_i = \exp \left( -\frac{|r - c^{-1(j)}|^2 - |r - c'j|^2}{2\sigma_n^2} \right) \text{ with } c' \in \{s^{-1(j)}\} \]  

(2.31)

For high SNR, as \( \sigma_n \to 0 \), \( \sum_i A_i \approx \sum_i B_i \), hence an approximation of \( \Lambda(d_j) \) can be obtained as

\[ \Lambda(d_j) \approx \Lambda'(d_j) = \frac{1}{2\sigma_n^2} \left( |r - c^{+1(j)}|^2 - |r - c^{-1(j)}|^2 \right) \]  

(2.32)
Expanding (2.32) using (2.22), we obtain

\[ \Lambda'(d_j) = \frac{2}{\sigma_n^2} \left( r_j + \sum_{i=1}^{n} r_i c_i^{A(j)} p_i \right) \]  \hspace{1cm} (2.33)

where

\[ p_i = \begin{cases} 0, & \text{if } c_i^{A(j)} = c_i^{B(j)} \\ 1, & \text{if } c_i^{A(j)} \neq c_i^{B(j)} \end{cases} \]  \hspace{1cm} (2.34)

Normalizing \( \Lambda'(d_j) \) with respect to the constant \( 2/\sigma_n^2 \), we obtain

\[ r'_j = \frac{\sigma_n^2}{2} \Lambda'_j = r_j + w_j \]  \hspace{1cm} (2.35)

where the extrinsic information is given by

\[ w_j = \sum_{i=1}^{n} r_i c_i^{A(j)} p_i \]  \hspace{1cm} (2.36)

### 2.5.2.2 Extrinsic Information

As shown in (2.32), computing the reliability of decoding decision \( d_j \) at the output of the soft-input decoder needs two code words \( c^{A(j)} \) and \( c^{B(j)} \). The soft decision \( d \) from Chase algorithm is one of these two code words. Assuming that another codeword \( c \) can be found from the candidate codewords \( \{c^i\} \) of the Chase algorithm to have minimum Euclidean distance from \( r \) with \( c_j \neq d_j \), it can be viewed as a competing code word of \( d \).

Given codeword \( c \) and \( d_j \), by normalizing (2.32) with respect to the constant \( 2/\sigma_n^2 \), the soft output is given by
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\[ r'_j = \left( \frac{|r - c|^2 - |r - d|^2}{4} \right) d_j \]  
\hspace{1cm} (2.37)

But in some cases, a competing codeword \( c \) cannot be found from \( \{c^t\} \). In this case, the size of the search space scanned by the Chase algorithm should be increased and the computing complexity will also increase. To alleviate this problem, Pyndiah proposed a simple solution for computing the soft output:

\[ r'_j = \beta d_j \]  
\hspace{1cm} (2.38)

where \( \beta \geq 0 \) can be manually obtained a-priori.

Therefore the extrinsic information can be expressed as

\[
    w_j = r'_j - r_j = \begin{cases} 
    \left( \frac{|r - c|^2 - |r - d|^2}{4} \right) d_j - r_j & \text{if } c \text{ exists} \\
    \beta d_j - r_j & \text{if } c \text{ does not exist}
    \end{cases}
\]  
\hspace{1cm} (2.39)

2.5.3 Turbo Decoding of Product Codes

Let us consider the decoding of the rows and columns of a product code transmitted on a Gaussian channel using BPSK signaling. On receiving the code matrix \([R]\) corresponding to a transmitted codeword \([E]\), the first decoder performs SISO decoding of the rows (or columns) of \([R]\) using the Chase algorithm. The soft output is computed using (2.37) and (2.38) while the extrinsic information \([W]\) is computed using (2.39). The soft input for the decoding of the next columns (or rows) is then given by

\[ [R(m+1)] = [R] + \alpha(m)[W(m)] \]  
\hspace{1cm} (2.40)
where the index \( m \) denotes the \( m^{th} \) decoding step, and \( \alpha \) is a scaling factor which takes into account the fact that the standard deviation of samples in \( [R] \) and \( [W] \) are different. The standard deviation of the extrinsic information is very high in the first decoding step and decreases as the decoding iterations proceed. This scaling factor \( \alpha \) is used to reduce the effect of the extrinsic information in the first decoding steps when the BER is relatively high. It takes a small value in the early decoding steps and increases as the BER improves. The component decoding procedure described above is depicted graphically in Fig. 2.5. It is cascaded for successive row/column decoding of the product code to perform iterative decoding.

The values of weighting factor \( \alpha \) used by Pyndiah at the \( m^{th} \) decoding step are given by

\[
\alpha(m) = [0.0, 0.2, 0.3, 0.5, 0.7, 0.9, 1.0, 1.0].
\] (2.41)

The values of reliability factor \( \beta \) used by Pyndiah at the \( m^{th} \) decoding are given by

\[
\beta(m) = [0.2, 0.4, 0.6, 0.8, 1.0, 1.0, 1.0, 1.0].
\] (2.42)
These $\alpha$ and $\beta$ values are used optimized manually for product codes composed of Hamming and BCH codes for AWGN and Rayleigh fading channels [1].

As from (2.41) and (2.42), the pre-computed sequences of $\alpha$ and $\beta$ allow for a maximum of 4 iterations (which we shall use in our simulations). For greater number of iterations, the adaptive versions of $\alpha$ and $\beta$ in [4][5][35]may be used.

2.6 Pyndiah's SISO Decoding Algorithm of Product Code for Rayleigh Fading Channel

The decoding process for product codes in a Rayleigh fading channel is similar to that in Section 2.5 except for small changes. Hence in this section, we shall only present the changes, assuming a coherently detected, fully interleaved Rayleigh fading channel.

2.6.1 Rayleigh Flat Fading Channel Model

$$r = x[A] + n$$

With the same definitions of $x$ and $n$ as in Section 2.5.1.1, Fig. 2.6 shows the model for a Rayleigh flat fading channel. Given the above channel, the received signal is given by

$$r = x[A] + n$$ (2.43)
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Where \( A \) is a diagonal matrix with the fading coefficient \( a_i \). Each \( a_i = a_i + j a_Q \) where \( a_i \) and \( a_Q \) are Gaussian distributed with their variance normalized to \( \frac{1}{2} \) such that the magnitude \( a_i = \sqrt{a_i^2 + a_Q^2} \) is Rayleigh distributed with \( E[|a_i|^2] = 1 \).

2.6.2 Decoding with Channel State Information

Assuming perfectly known \( A \) at the receiver, as opposed to (2.19), the log-likelihood ratio (LLR) of decision \( r_j \) of the Rayleigh fading channel is given by

\[
\Lambda(r_j) = \ln \frac{\Pr\{x_j = +1, a_j | r_j\}}{\Pr\{x_j = -1, a_j | r_j\}}
\]

\[
= \ln \frac{\Pr\{r_j | x_j = +1, a_j\}}{\Pr\{r_j | x_j = -1, a_j\}}
\]

\[
= \frac{2}{\sigma_n^2} a_j r_j
\]

where

\[
\Pr\{r_j | x_j, a_j\} = \frac{1}{2\pi \sigma_n^2} \exp \left( -\frac{r_j - a_j x_j}{2\sigma_n^2} \right).
\]

It can be shown that reliability of decision \( d_j \) at the output of the decoder, as defined using the log-likelihood ratio (LLR), can be approximated by

\[
\Lambda(d_j) \approx \Lambda'(d_j) = \frac{1}{2\sigma_n^2} (|r - c^{(1)}[A]|^2 - |r - c^{(2)}[A]|^2)
\]
Expanding (2.46) using $|r - c'[A]|^2 = \sum_{i=1}^{n} (r_i - c'_i a_i)^2$, we obtain

$$\Lambda'(d_j) = \frac{2}{\sigma_n^2} \left( a_j r_j + \sum_{i=1, i \neq j}^{n} a_i r_i c_i^{(j)} p_i \right)$$

(2.47)

where

$$p_i = \begin{cases} 0, & \text{if } c_i^{(j)} = c_i^{(j)} \\ 1, & \text{if } c_i^{(j)} \neq c_i^{(j)} \end{cases}$$

(2.48)

Normalizing $\Lambda'(d_j)$ with respect to the constant $2/\sigma_n^2$, we obtain

$$r'_j = a_j r_j + w_j$$

(2.49)

where the extrinsic information is given by

$$w_j = \sum_{i=1, i \neq j}^{n} a_i r_i c_i^{(j)} p_i$$

(2.50)

Also, in contrast with (2.37), the soft output of the decoder in a Rayleigh fading channel is given by

$$r'_j = \left( \frac{|r - c[A]|^2 - |r - d[A]|^2}{4} \right) d_j .$$

(2.51)

where both $c[A]$ and $d[A]$ have the minimum Euclidean distance from $r$ from the set of candidates codewords $\{c^*\}$ and that $c_j \neq d_j$.

It is interesting to note that if the fading coefficient matrix $[A]$ is multiplied on $r$ instead of on $c$ and $d$, the same soft output as (2.47) will result:
\[ \Lambda(d_j) \approx \Lambda'(d_j) = \frac{1}{2\sigma_n^2} (|r[A] - c^{(j)}|^2 - |r[A] - c^{-1}(j)|^2) = \frac{2}{\sigma_n^2} \left( a_j r^j + \sum_{l \neq j} a_l r^l c^{(l)}_j p_l \right) \]

Therefore, the same soft output of the decoder can also be found by

\[ r^j = \frac{|r[A] - c|^2 - |r[A] - d|^2}{4} d_j \] (2.50)

### 2.6.3 Decoding Without Channel State Information

When \([A]\) is not known at the receiver, the log-likelihood ratio (LLR) of decision \(r_j\) of the Rayleigh fading channel is given by (2.19) and the soft output of the decoder is given by (2.37).
Chapter 3

Effect of Code Shortening on Decoding BER Performance

As shown earlier in Section 2.4, the weight distribution of a code enables the prediction of BER through the union bounds. However, when a code is shortened, its weight distribution will change and its BER performance will be affected. Hence, in this section, we shall derive the approximated weight enumerator for a shortened code and thereafter apply this approximated weight enumerator to the BER bounds.

Note that in this chapter we refer to the conventional way of code shortening as described in Section 2.3.2.

3.1 Weight Enumerator of Shortened Code

3.1.1 Approximation to the Weight Distribution of Shortened Binary Linear Codes

As mentioned in Section 2.1.4, it has been observed that the binomial distribution is a close approximation to the weight distributions of most binary \((n, k)\) linear codes. For a
shortened code with \( s \) number of nullified information symbols, we also assume that its weight distribution is binomial-like (to be verified later).

Following (2.1) and denoting the maximum Hamming distance of a shortened code as \( d_{\text{max}} \), the weight enumerator of a shortened code \( A_d(s) \) is then given by

\[
A_d(s) = \begin{cases} 
1, & d = 0 \\
\delta \binom{n-s}{d}, & d_{\text{min}} \leq d \leq d_{\text{max}} \\
0, & \text{elsewhere}
\end{cases}
\]  

(3.1)

where

\[
\delta = \frac{2^{k-s} - 1}{\sum_{d=d_{\text{min}}}^{d_{\text{max}}} \binom{n-s}{d}}.
\]  

(3.2)

In (3.2), \( \delta \) is chosen such that \( \sum_d A_d(s) = 2^{k-s} \) because the binomial approximation is applied to all its \( 2^{k-s} \) shortened codewords except for the all-zero codeword.

**Example**

To illustrate with a simple example, the weight distribution from (3.1) for a shortened-Hamming (6, 3) is given by \( A_0 = 1, A_3 = 4 \) and \( A_4 = 3 \) with \( \delta = 0.2 \). As shown in Fig. 3.1, this estimated weight distribution agrees exactly with its true weight distribution. Note that as the shortened code does not contain the all-one codeword, the weight distribution is no longer palindromic (i.e. \( A_d \neq A_{n-d} \)).
Chapter 3 - Effect of Code Shortening on Decoding BER Performance

Distance and Weight Enumerator of $s$-Hamming(6, 3) with $s = 1$

Binomial Appr.  True

Fig. 3.1. Comparison of Approximated $A_d$ and true $A_d$ of $s$-Hamming(6, 3) with $s = 1$

Fig. 3.2 and 3.3 show the comparison of $A_d$ obtained from the binomial approximation (3.1) with its true $A_d$ for shortened-BCH(28, 13) with $s = 3$ and shortened-Hamming(25, 20) with $s = 6$ respectively. Both figures show that (3.1) gives a reasonable approximation of the weight distribution of the shortened codes. They also validate our original assumption that shortened codes have binomial-like weight distribution as (3.1).
Chapter 3 – Effect of Code Shortening on Decoding BER Performance

Fig. 3.2. Comparison of Approximated $A_d$ and true $A_d$ of s-BCH(28,13) with $s=3$

Fig. 3.3. Comparison of Approximated $A_d$ and true $A_d$ of s-Hamming(25,20) with $s=6$
Chapter 3 - Effect of Code Shortening on Decoding BER Performance

It would be useful to express $A_{d(s)}$ in terms of $A_d$ as this will allow us to compute easily the weight distribution of the shortened codes based on the known weight distributions of the un-shortened codes.

We define a scaling factor

$$f_{s,d} = \frac{A_{d(s)}}{A_d}.$$  \hspace{1cm} (3.3)

Using (2.1) and (3.1), $f_{s,d}$ can then be expressed as

$$f_{s,d} = \begin{cases} \frac{1}{2^{s'-s}}, & \text{if } d=0 \\ \left(\begin{array}{c} n-s \\ d \end{array}\right) \sum_{d=d_{\min}}^{d_{\max}} \left(\begin{array}{c} n \\ d \end{array}\right), & \text{if } (n-s) \geq d \\ 0, & \text{if } (n-s) < d \end{cases}$$ \hspace{1cm} (3.4)

where $n$ and $d$ are parameters of the original (un-shortened) code and $s$ is the number of nullified information symbols.

Define the set of code distances which excludes the $d = 0$ and $d = n$ terms of an un-shortened code to be \{d_{\text{atj}}\}. Then

$$\sum_{d=d_{\text{at}}}^{d_{\max}} \left(\begin{array}{c} n \\ d \end{array}\right) = \sum_{d\in\{d_{\text{at}}\}} \left(\begin{array}{c} n \\ d \end{array}\right).$$ \hspace{1cm} (3.5)

Making use of the known result [33]

$$\sum_{d=d_{\text{at}}}^{d_{\max}} \left(\begin{array}{c} n \\ d \end{array}\right) = 2^n,$$ \hspace{1cm} (3.6)

we have
\[ \sum_{d \in \{d_{\text{sh}}\}} \binom{n}{d} = 2^n - \sum_{d \in \{d\}} \binom{n}{d} \quad (3.7) \]

Similarly, we define the set of code distances which excludes the \( d = 0 \) term of a shortened code to be \( \{d_{\text{sh}}\} \). Then

\[ \sum_{d = d_{\text{min}}}^{d_{\text{max}}} \binom{n-s}{d} = \sum_{d \in \{d_{\text{sh}}\}} \binom{n-s}{d} \quad (3.8) \]

Similarly, making use of (3.6), we obtain

\[ \sum_{d \in \{d_{\text{sh}}\}} \binom{n-s}{d} = 2^{n-s} - \sum_{d \in \{d\}} \binom{n-s}{d} \quad (3.9) \]

Therefore,

\[ \frac{n-d_{\text{min}}}{n-d_{\text{max}}} \binom{n}{d} = \frac{2^n - \sum_{d \in \{d\}} \binom{n}{d}}{\sum_{d = d_{\text{min}}}^{d_{\text{max}}} \binom{n-s}{d}} = 2^{n-s} - \sum_{d \in \{d_{\text{sh}}\}} \binom{n-s}{d} \]

\[ = \left(1 - \frac{\sum_{d \in \{d_{\text{sh}}\}} \binom{n}{d}}{2^n}\right) \frac{1}{2^{n-s}} \quad (3.10) \]

When \( n \) is large, \( \frac{\sum_{d \in \{d_{\text{sh}}\}} \binom{n}{d}}{2^n} \to 0 \) and \( \frac{\sum_{d \in \{d\}} \binom{n-s}{d}}{2^{n-s}} \to 0 \).

Therefore, (3.10) is approximately equal to
Chapter 3 – Effect of Code Shortening on Decoding BER Performance

\[ \sum_{d=s}^{n-d_{\text{max}}} \left( \begin{array}{c} n \\ d \end{array} \right) \approx \frac{2^n}{2^{n-s}} = 2^{n-s} \] \hspace{1cm} (3.11)

With (3.11), and since \( n \) and \( k \) are usually large, (3.4) can be simplified to

\[ f_{s,d} = \begin{cases} 1 & \text{if } d = 0 \\ \left( \begin{array}{c} n-s \\ d \end{array} \right) & \text{if } (n-s) \geq d \\ \left( \begin{array}{c} n \\ d \end{array} \right) & \text{if } (n-s) < d \end{cases} \] \hspace{1cm} (3.12)

Therefore,

\[ A_{d(s)} = \text{Round} \left( f_{s,d} A_d \right) \] \hspace{1cm} (3.13)

where \( \text{Round}(.) \) refers to rounding of the value(.) to the nearest integer.

### 3.1.2 Comparison of True and Approximated weight enumerator of Shortened Code

Tables 3.1, 3.2 and 3.3 compare the true and approximated \( A_{d(s)} \) values for shortened Golay, Hamming and BCH codes with different amount of shortening \( s \). Disagreements between the true and approximated values are highlighted in bold italic. We can see that there is no difference in the approximated and true \( A_{d(s)} \) for shortened Hamming and BCH codes with \( s = 1 \) and 2 and, for shortened Golay codes \( s = 1 \) to 7 and \( s = 9 \). These results show that (3.13) gives tight estimates to the true \( A_{d(s)} \). More comparisons for larger Hamming and BCH codes are shown in Appendices A and B. Similarly, they show that
(3.13) gives good estimate to the true $A_{d(s)}$.

Table 3.1. Comparison of True and Approximated $A_{d(s)}$ for Golay code

<table>
<thead>
<tr>
<th>Golay(23, 12)</th>
<th>$d$</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 7 8 11 12 15 16 23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1-7 True $A_{d(s)}$</td>
<td>Agree Exactly</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approx. $A_{d(s)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 True $A_{d(s)}$</td>
<td>1 7 6 1 1 0 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approx. $A_{d(s)}$</td>
<td>1 7 7 1 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 True $A_{d(s)}$</td>
<td>Agree Exactly</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approx. $A_{d(s)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3.2. Comparison of True and Approximated $A_d(s)$ for Hamming code

<table>
<thead>
<tr>
<th>Hamming(7, 4)</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$s$</td>
<td>1-2</td>
</tr>
<tr>
<td>True $A_d(s)$</td>
<td></td>
</tr>
<tr>
<td>Approx. $A_d(s)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hamming(15, 11)</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$s$</td>
<td>1-2</td>
</tr>
<tr>
<td>True $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>Approx. $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>True $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>Approx. $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>True $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>Approx. $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>True $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>Approx. $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>True $A_d(s)$</td>
<td>1</td>
</tr>
<tr>
<td>Approx. $A_d(s)$</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: Agree Exactly indicates that the true and approximated values of $A_d(s)$ are identical for each $d$. The table shows the comparison for Hamming codes of different lengths.
Table 3.3. Comparison of True and Approximated $A_d(s)$ for BCH code

<table>
<thead>
<tr>
<th>$s$</th>
<th>1-2</th>
<th>3-5</th>
<th>Agree Exactly</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True $A_d(s)$</td>
<td>Approx. $A_d(s)$</td>
<td></td>
</tr>
<tr>
<td>1-2</td>
<td>1 5  5  2</td>
<td>1 5  6  2</td>
<td>Agree Exactly</td>
</tr>
<tr>
<td>3</td>
<td>1 5  5  2</td>
<td>1 5  6  2</td>
<td>Agree Exactly</td>
</tr>
<tr>
<td>4-5</td>
<td>True $A_d(s)$</td>
<td>Approx. $A_d(s)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s$</th>
<th>1-2</th>
<th>3-5</th>
<th>Agree Exactly</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True $A_d(s)$</td>
<td>Approx. $A_d(s)$</td>
<td></td>
</tr>
<tr>
<td>1-2</td>
<td>1 70 183 1320 1872 2276 1847 424 192 6 1 0</td>
<td>1 70 1321 1871 2275 1848 425 191 6 1 0</td>
<td>Agree Exactly</td>
</tr>
<tr>
<td>3</td>
<td>1 70 183 1321 1871 2275 1848 425 191 6 1 0</td>
<td>1 70 183 1321 1871 2275 1848 425 191 6 1 0</td>
<td>Agree Exactly</td>
</tr>
<tr>
<td>4</td>
<td>1 53 130 800 1072 1058 789 136 58 1 0 0</td>
<td>1 52 131 802 1069 1056 792 137 55 1 0 0</td>
<td>Agree Exactly</td>
</tr>
<tr>
<td>5</td>
<td>1 40 90 472 600 472 317 40 16 0 0 0</td>
<td>1 39 92 475 594 469 323 40 14 0 0 0</td>
<td>Agree Exactly</td>
</tr>
<tr>
<td>6</td>
<td>1 27 63 279 321 193 124 13 3 0 0 0</td>
<td>1 28 64 274 320 199 124 11 3 0 0 0</td>
<td>Agree Exactly</td>
</tr>
<tr>
<td>7</td>
<td>1 18 42 162 168 70 45 6 0 0 0 0</td>
<td>1 20 43 154 166 79 45 3 1 0 0 0</td>
<td>Agree Exactly</td>
</tr>
</tbody>
</table>
3.2 Weight Enumerator of Punctured Code

In Section 3.1, we have discussed and shown the effect of code shortening on decoding BER performance. In particular, we show that the weight enumerator of shortened code can be obtained from (3.13) with the use of (3.12). (3.13) can be interpreted as: The ratio of codewords with hamming distance $d$ to the $n$-tuples with hamming distance $d$ before shortening is approximately the same as the ratio of codewords with hamming distance $d$ to the $(n-s)$-tuples with hamming distance $d$ after shortening, i.e. $\frac{A_d}{n_d} = \frac{A_{d,0}}{n-s}$. Unlike code shortening where information symbols are nullified, puncturing instead nullifies the parity symbols. In this section, we shall attempt to find the weight enumerator of a punctured code using the afore-mentioned result.
3.2.1 Distance Distribution of Punctured Codes

Consider the case when a single parity bit is being punctured. This parity bit can take the value of either ‘1’ or ‘0’. Hence the hamming distance can either remain as before or decrease by 1. Fig. 1 shows the true distance and weight distributions of punctured codes from original code: Hamming(15,11,3) with \( p = 1 \), \( 2 \) and \( 3 \) where \( p \) is the number of punctured parity bits. Fig. 1 shows that when \( p \) increases by a single bit, hamming distance generally decreases by 1. For example, when \( p = 1 \), the minimum hamming distance decreases from 3 to 2. And when \( p \) is increased by 1 again (to 2), the minimum hamming distance remains at 2.

3.2.2 Approximated Weight Enumerator for Punctured Code with Single Punctured Parity Bit

From Fig. 3.4, similarly to shortened code, it can be observed that the weight distribution of punctured code is binomial-like. Fig. 3.5 shows the true distance and weight distributions of punctured-BCH(p-BCH) (8,7) with \( p = 7 \). It shows that the weight distribution of the punctured code is still binomial-like. Hence we shall apply (3.12) and (3.13), given the hamming distance variation when puncturing is performed (Section (3.2.1)), to find the weight enumerator of punctured code.

Consider the case when the number of punctured parity bit is 1 i.e. \( p = 1 \). Let \( A_{d(p)} \) be the number of codewords of hamming distance \( d \) for a punctured code. Since puncturing of a single parity bit will cause the Hamming distance of the codewords of the original code to either remain the same (if the parity bit happens to be zero) or decrease by 1 (if
the parity bit happens to be 1), the contribution of the codewords to $A_{d(p)}$ of the punctured code can be due to $A_d$ or $A_{d+1}$ of the original code.

Therefore using the "proportion factor" given in (3.12), the weight enumerator for a punctured code can be approximated by

$$A_{d(p)} = \begin{cases} \text{Round} \left( \frac{n-1}{d} A_d \right) + \text{Round} \left( \frac{n-1}{d+1} A_{d+1} \right), & \text{if } d \leq (n-p) \\ 0, & \text{elsewhere} \end{cases}$$

(3.14)

Fig. 3.4. True distance and weight distributions of punctured codes from Hamming(15,11,3) with $p=1,2,3$
Fig. 3.5. True distance and weight distributions of punctured-Hamming(8,7) with \( p = 7 \)

Table 3.4 compares the true and the approximated \( A_{d(p)} \) (1) for \( p\)-Hamming(6,4). It shows that the true and approximated weight enumerator agree exactly.

Table 3.4. Comparison of True and Approximated \( A_{d(p)} \) for \( p\)-Hamming(6,4)

<table>
<thead>
<tr>
<th>( p = 1 )</th>
<th>( d )</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>True ( A_{d(p)} )</td>
<td>1 #</td>
<td>3</td>
<td>8</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Approx. ( A_{d(p)} )</td>
<td>1 #</td>
<td>3</td>
<td>8</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
3.2.3 Approximated Weight Enumerator with Numerous Punctured Parity Bits

The weight enumerator of (3.14) has been obtained based on the assumption that \( p = 1 \). However, when \( p \) is larger than 1, (3.14) can be used repeatedly till the number of punctured parity bits \( p \) is attained. As an illustration, following the previous example, the distance and weight distribution of punctured code \( p\text{-Hamming}(5, 4) \) (with \( p = 2 \)) can be obtained from the distance and weight distribution of punctured code \( (6, 4) \) using (3.14). Hence in order to compute the weight distribution of \( p\text{-Hamming}(5, 4) \) (with \( p = 2 \)) from original code \( \text{Hamming}(7, 4) \), (3.14) is used twice: first to compute the weight distribution of \( p\text{-Hamming}(6, 4) \) from \( \text{Hamming}(7, 4) \) and then the weight distribution of \( p\text{-Hamming}(5, 4) \) from \( p\text{-Hamming}(6, 4) \).

3.2.4 Comparison of True and Approximated Weight Enumerators of Punctured Codes

Tables 3.5-3.8 compares the true and approximated weight enumerator of punctured codes from: \( \text{Hamming}(7,4,3) \) (Table 3.5), \( \text{Hamming}(15,11,3) \) (Table 3.6), \( \text{BCH}(15,7,5) \) (Table 3.7) and \( \text{BCH}(15,5,7) \) (Table 3.8). It has been observed that when \( p \) is small (\( p = 1-2 \)), the true and approximated weight enumerators agree exactly. However, when \( p \) is large (beyond 2), it has been observed that the approximated weight enumerator is not a good estimate.
Table 3.5. Comparison of True and Approximated $A_{d,p}$ for Hamming(7,4,3)

<table>
<thead>
<tr>
<th>$d$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>1-2</td>
<td>True $A_{d,p}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Approx. $A_{d,p}$</td>
<td>Agree Exactly</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Table 3.6. Comparison of True and Approximated $A_{d,p}$ for Hamming(15,11,3)

<table>
<thead>
<tr>
<th>$d$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>1-2</td>
<td>True $A_{d,p}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Approx. $A_{d,p}$</td>
<td>Agree Exactly</td>
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</tr>
<tr>
<td></td>
<td>3</td>
<td>True $A_{d,p}$</td>
<td>1</td>
<td>4</td>
<td>34</td>
<td>116</td>
<td>239</td>
<td>392</td>
<td>476</td>
<td>392</td>
<td>239</td>
<td>116</td>
<td>34</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Approx. $A_{d,p}$</td>
<td>1</td>
<td>4</td>
<td>34</td>
<td>117</td>
<td>238</td>
<td>391</td>
<td>478</td>
<td>391</td>
<td>238</td>
<td>117</td>
<td>34</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 3.7. Comparison of True and Approximated $A_d(p)$ BCH(15,7,5)

| BCH(15,7,5) | \(d\) | \\hline | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **True \(A_d(p)\)** | \(1\) | \(0\) | \(0\) | \(2\) | \(12\) | \(27\) | \(22\) | \(22\) | \(27\) | \(12\) | \(2\) | \(0\) | \(0\) | \(1\) |
| **Approx. \(A_d(p)\)** | \(1\) | \(0\) | \(0\) | \(2\) | \(13\) | \(26\) | \(22\) | \(22\) | \(26\) | \(13\) | \(2\) | \(0\) | \(0\) | \(1\) |
| **True \(A_d(p)\)** | \(1\) | \(0\) | \(1\) | \(4\) | \(19\) | \(28\) | \(22\) | \(28\) | \(19\) | \(4\) | \(1\) | \(0\) | \(1\) | \(0\) |
| **Approx. \(A_d(p)\)** | \(1\) | \(0\) | \(0\) | \(6\) | \(19\) | \(26\) | \(24\) | \(26\) | \(19\) | \(6\) | \(0\) | \(0\) | \(1\) | \(0\) |
| **True \(A_d(p)\)** | \(1\) | \(0\) | \(3\) | \(6\) | \(29\) | \(25\) | \(25\) | \(29\) | \(6\) | \(3\) | \(0\) | \(1\) | \(0\) | \(0\) |
| **Approx. \(A_d(p)\)** | \(1\) | \(0\) | \(2\) | \(11\) | \(24\) | \(27\) | \(27\) | \(24\) | \(11\) | \(2\) | \(0\) | \(1\) | \(0\) | \(0\) |
| **True \(A_d(p)\)** | \(1\) | \(1\) | \(3\) | \(16\) | \(28\) | \(30\) | \(28\) | \(16\) | \(3\) | \(1\) | \(1\) | \(0\) | \(0\) | \(0\) |
| **Approx. \(A_d(p)\)** | \(1\) | \(0\) | \(5\) | \(17\) | \(28\) | \(29\) | \(29\) | \(17\) | \(5\) | \(0\) | \(1\) | \(0\) | \(0\) | \(0\) |
Table 3.8. Comparison of True and Approximated $A_{d(p)}$ for BCH(15,5,7)

<table>
<thead>
<tr>
<th>BCH(15,5,7)</th>
<th>$d$</th>
<th>Agree Exactly</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>0</td>
<td>True</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>0</td>
</tr>
</tbody>
</table>

| 3           | 0   | True | Approx. |
|             | 1   | 0    | 0       |
|             | 2   | 0    | 0       |
|             | 3   | 1    | 7       |
|             | 4   | 13   | 7       |
|             | 5   | 1    | 0       |
|             | 6   | 0    | 0       |
|             | 7   | 1    | 0       |
|             | 8   | 0    | 0       |
|             | 9   | 1    | 0       |
|             | 10  | 0    | 0       |
|             | 11  | 1    | 0       |
|             | 12  | 0    | 0       |

| 4           | 0   | True | Approx. |
|             | 1   | 0    | 0       |
|             | 2   | 0    | 0       |
|             | 3   | 0    | 4       |
|             | 4   | 11   | 4       |
|             | 5   | 1    | 0       |
|             | 6   | 0    | 0       |
|             | 7   | 1    | 0       |
|             | 8   | 0    | 0       |
|             | 9   | 1    | 0       |
|             | 10  | 0    | 0       |
|             | 11  | 1    | 0       |
|             | 12  | 0    | 0       |

| 5           | 0   | True | Approx. |
|             | 1   | 0    | 0       |
|             | 2   | 0    | 0       |
|             | 3   | 0    | 1       |
|             | 4   | 8    | 12      |
|             | 5   | 8    | 1       |
|             | 6   | 0    | 0       |
|             | 7   | 1    | 0       |
|             | 8   | 0    | 0       |
|             | 9   | 1    | 0       |
|             | 10  | 0    | 0       |
|             | 11  | 1    | 0       |
|             | 12  | 0    | 0       |

| 6           | 0   | True | Approx. |
|             | 1   | 0    | 0       |
|             | 2   | 0    | 0       |
|             | 3   | 0    | 4       |
|             | 4   | 11   | 4       |
|             | 5   | 1    | 0       |
|             | 6   | 0    | 0       |
|             | 7   | 1    | 0       |
|             | 8   | 0    | 0       |
|             | 9   | 1    | 0       |
|             | 10  | 0    | 0       |
|             | 11  | 1    | 0       |
|             | 12  | 0    | 0       |

| 7           | 0   | True | Approx. |
|             | 1   | 0    | 0       |
|             | 2   | 0    | 0       |
|             | 3   | 0    | 1       |
|             | 4   | 8    | 12      |
|             | 5   | 8    | 1       |
|             | 6   | 0    | 0       |
|             | 7   | 1    | 0       |
|             | 8   | 0    | 0       |
|             | 9   | 1    | 0       |
|             | 10  | 0    | 0       |
|             | 11  | 1    | 0       |
|             | 12  | 0    | 0       |

| 8           | 0   | True | Approx. |
|             | 1   | 0    | 0       |
|             | 2   | 0    | 0       |
|             | 3   | 0    | 4       |
|             | 4   | 11   | 4       |
|             | 5   | 1    | 0       |
|             | 6   | 0    | 0       |
|             | 7   | 1    | 0       |
|             | 8   | 0    | 0       |
|             | 9   | 1    | 0       |
|             | 10  | 0    | 0       |
|             | 11  | 1    | 0       |
|             | 12  | 0    | 0       |
3.3 BER Bounds of Shortened Codes in AWGN Channel

In Section 3.1, we have shown that the approximated weight enumerator for shortened code is a good estimate of the true weight enumerator. However the same is not true for the approximated weight enumerator of punctured code (Section 3.2). Hence for the following sections, we shall only focus on the BER bounds for shortened codes.

3.3.1 BER Bound of Shortened Block Code in AWGN Channel

With code-rate $r_{(s)} = \frac{k-s}{n-s}$ and weight enumerator of a shortened code given by (3.13), the decoding bit-error probability $P_{b(s)}$ of a shortened code in an AWGN channel can be obtained from (2.7):

$$P_{b(s)} \leq \sum_{d} \frac{d}{n-s} f_s d^A P_d \left( d, r_{(s)} \right)$$

(3.15)

assuming that the shortened code has the set of Hamming distance $\{d_{oo}\}$ from the original code.

3.3.2 BER Bound for Shortened Product code in AWGN Channel

Consider a 2-D product code (PC) $C^p(n^p, k^p, d^p)$ consisting of horizontal linear block code $C^h(n^h, k^h, d^h)$ whose weight enumerator is $A^h$ and vertical linear block code $C^v(n^v, k^v, d^v)$ whose weight enumerator is $A^v$. $C^p$ has minimum Hamming distance $d^p = d^hd^v$, information length $k^p = k^hv^p$, code length $n^p = n^hn^v$ [1] and the number of minimum weight code-words of $C^p$ is given by the product of the number of minimum weight codewords...
of the component codes $A_n^h, A_d^v$ [3]. As the weight distribution of the product code is not easily known, and since the BER is mostly determined by the $d_{\min}$ term at higher $E_b/N_0$, we may lower-bound the BER by using only the $d_{\min}$ term in (2.7). This bound is known as the asymptotic BER bound [3].

Similarly, now consider a 2-D shortened-product code(s-PC) made up of shortened component codes $C^h_s\left(n^h - s^h, k^h - s^h, d^h\right)$ and $C^v_s\left(n^v - s^v, k^v - s^v, d^v\right)$. Assuming the minimum Hamming distance unchanged, the asymptotic lower BER bound of a 2-D shortened-product code(s-PC) made up of shortened component codes $C^h_s\left(n^h - s^h, k^h - s^h, d^h\right)$ and $C^v_s\left(n^v - s^v, k^v - s^v, d^v\right)$ is then given by

$$P_{h(s-PC)} \geq \frac{d^h d^v}{(n^h - s^h)(n^v - s^v)} f_{h,d}^h A_n^h A_d^v P_e(d^h d^v, r_{s-PC}^e)$$

(3.16)

where $r_{s-PC}^e$ is the code-rate of the s-PC, and $f_{h,d}^h$ and $f_{v,d}^v$ scaling factors for the component codes are given by (3.12).

### 3.3.3 BER Bound for Extended-Shortened Codes

Extended product code (e-PC) refers to a product code made up of extended horizontal and/or vertical component codes. Shortened-extended product code (s-e-PC) refers to a product code made up of extended horizontal and/or vertical component codes which have been shortened.

With code-rate of s-e-PC as $r_{e}^{s-PC}$, the asymptotic bounds of s-e-PC in an AWGN channel is given by

$$P_{s(e-PC)} \geq \frac{(d^h + 1)(d^v + 1)}{(n^h + 1 - s^h)(n^v + 1 - s^v)} f_{h,d+1}^h f_{v,d+1}^v f_{h+1}^{d} f_{v+1}^{d} P_e\left((d^h + 1)(d^v + 1), r_{e}^{s-PC}\right)$$

(3.17)
Chapter 3 - Effect of Code Shortening on Decoding BER Performance

where

\[ A^{\text{h}}_{d+1} = A^h_d + A^h_{d+1} \]

is the number of codewords of weight \( d^h + 1 \) for the extended horizontal component code (see Section 2.3.1),

\[ A^{\text{v}}_{d+1} = A^v_d + A^v_{d+1} \]

is the number of weight \( d^v + 1 \) for the extended vertical component code (see Section 2.3.1), and \( P_d \left( (d^h + 1)(d^v + 1), r^{\text{e-PC}}_c \right) \) is given in (2.8).

3.4 BER Bounds of Shortened Codes in Rayleigh Fading Channel

The asymptotic BER bounds for shortened codes in a Rayleigh fading channel can be easily obtained by replacing the pair-wise error probability of the AWGN channel (2.8) by that of the Rayleigh fading channel (2.16).

Specifically,

- the asymptotic BER bound of shortened block code in a Rayleigh fading channel can be obtained from (3.15) by replacing \( P_d \left( d, r_{(c)} \right) \) by \( \overline{P}_d \left( d, r_{(c)} \right) \) and using the \( d_{\text{min}} \) term only;

- the asymptotic BER bound of s-PC in a Rayleigh fading channel can be obtained from (3.16) by replacing \( P_d \left( d^h d^v s^{(\text{PC})}_c \right) \) by \( \overline{P}_d \left( d^h d^v s^{(\text{PC})}_c \right) \);

- the asymptotic BER bound of s-e-PC in a Rayleigh fading channel can be obtained from (3.17) by replacing \( P_d \left( (d^h + 1)(d^v + 1), r^{\text{e-PC}}_c \right) \) by

\[ \overline{P}_d \left( (d^h + 1)(d^v + 1), r^{\text{e-PC}}_c \right). \]
3.5 Asymptotic Coding Loss of Shortened Codes

As code shortening typically results in poorer BER, a shortened code typically requires more SNR than an unshortened code to achieve the same decoding bit error probability. This additional SNR, expressed in decibels, required by a shortened code to achieve a specific bit error probability as the unshortened code is termed *coding loss*. The asymptotic coding loss is obtained when the asymptotic BER bounds of the unshortened code and shortened code are used to find the coding loss. In this section, we derive the generic coding loss expressions when a code is shortened.

The asymptotic coding loss in an AWGN channel between a shortened code and an un-shortened can be estimated by

1. using the asymptotic BER bound and,
2. bounding $Q(x)$ in (2.8) by $\frac{1}{2}\exp(-x^2/2)$ [23].

Using the scaling factor $f_{s,d}$ from (3.12) and with $d_{\text{min}}$ of shortened code unchanged, the coding loss $L_{\text{awgn}}$ in an AWGN channel is given by

$$L_{\text{awgn}}(\text{dB}) = 10\log_{10} \left( \frac{r_c}{r_{c(s)}} \right) + \frac{\ln \left( \frac{n}{n-s} \right) + \ln f_{\text{s,d}}}{d_{\text{min}} r_{c(s)} E_b / N_0} \right) \quad (3.18)$$

Note that if there is truly a coding loss after code shortening, (3.18) will give positive values, otherwise the Log(.) term will give negative values, which imply coding gain instead of coding loss after code shortening. Assuming large $E_b / N_0$, (3.18) becomes $10\log_{10} \left( \frac{r_c}{r_{c(s)}} \right)$, which is always positive because $r_c$ is always larger than $r_{c(s)}$ and hence a coding loss.
In Rayleigh fading channel, the coding loss is similarly estimated by

1. using the asymptotic BER bound and,

2. approximating $P_d(d, r_c)$ by $P_d(d, r_c) \leq \left[ \frac{1}{2 \cdot 1 + r_c \frac{E_b}{N_0}} \right]^d$ \[26\].

Using scaling factor $f_{s,d}$ from (3.12) and with $d_{\text{min}}$ of shortened code unchanged, the coding loss $L_{\text{Rayleigh}}$ in a Rayleigh fading channel is then given by

$$L_{\text{Rayleigh}}(dB) = 10 \log_{10} \left( \frac{G - 1}{r_{c(s)}} + \frac{Gr_c}{Gr_{c(s)}} \right)$$ \[3.19\]

where

$$G = \left( \frac{n}{n - s f_{s,d_{\text{min}}}} \right)^{d_{\text{min}}}$$ \[3.20\]

Similarly, (3.19) will give positive values for coding loss and negative values for coding gain. Assuming large $E_b/N_0$, (3.19) becomes $10 \log_{10} \left( \frac{Gr_c}{Gr_{c(s)}} \right)$ with $\frac{n}{n - s f_{s,d_{\text{min}}}} > 1$, $0 < f_{s,d_{\text{min}}} < 1$ and $d_{\text{min}}^{-1} < 1$. So depending on the code parameters $(r_c, n, d_{\text{min}})$ and the amount of shortening $(s)$, coding gain or coding loss may result.
3.6 Simulations and Discussions

In this section, we will verify the tightness of the BER bounds developed in the previous sections for shortened block codes, shortened product codes and shortened-extended product codes using computer simulations and other benchmarks. In the following discussions, shortened BCH code is abbreviated as s-BCH and shortened Hamming code as s-Hamming.

3.6.1 Results for BER Bounds

Fig. 3.6 shows the simulated BER and the BER union bound values for s-Hamming(6, 3) and s-BCH(14, 6) with 1 nullified bit, i.e. \( s = 1 \). The simulated BER is obtained by Monto Carlo simulations of maximum-likelihood soft decoding using all possible codewords, i.e. brute-force decoding. The BER bound is obtained from (3.15). The results in the figure show that the bound (3.15) obtained using the weight enumerator scaling factor \( f_{s, d} \) in (3.12) is tight and matches the simulated BER values very closely at mid-to-high \( E_b/N_0 \) values. At low \( E_b/N_0 \) values, the bound is looser and lie above the simulated BER line because union bound is by nature an overbound (upper bound).
Fig. 3.6. Proposed BER bounds and simulated BER values for Hamming(7, 4) and BCH(15, 7) shortened by 1 bit (s=1) in AWGN channel.

Fig. 3.7 shows the simulated BER and its BER union bound for BCH(31, 16) with more shortened symbols (s=8 and s=10). The results again show that, notwithstanding the larger s, the bound obtained in (3.15) is tight for BER < 10^{-3}.
Fig. 3.7. Proposed BER bounds and simulated BER values for BCH(31,16) shortened by $s = 8$ and $s = 10$ in AWGN channel.

Fig. 3.8 shows the simulated BER and asymptotic lower bound in AWGN and Rayleigh fading channels for the shortened-extended product code s-e-PC(11,6,4)$^2$, which is obtained from the extended product code e-PC(16,11,4)$^2$ by shortening the first 5 bits in all rows and columns ($s^h = s^v = 5$). As this code is large, its simulated BER results are obtained using Pyndiah's decoding algorithm (described in Sections 2.5 and 2.6) with 6 test positions (i.e. 64 test patterns) and 4 decoding iterations. More test positions and decoding iterations do not improve the performance further. The results again show that the proposed tight asymptotic bounds (3.17) match well with the true BER performance.
Fig. 3.8. Proposed asymptotic BER bounds and simulated BER values for $s$-e-PC$(11,6)^2$ shortened by $s = s' = 5$ in AWGN and Rayleigh fading channels
Chapter 4

Ordered Shortening

In this chapter, we propose a novel code shortening scheme, called Ordered Shortening, which carefully selects specific information symbol positions to shorten, instead of shortening a block of consecutive information symbol positions as in the conventional code shortening scheme, in order to obtain better code properties and decoding performance. Before we go into details, we shall explain the motivation behind this work.

4.1 Motivation

Table 4.1 compares the $d_{\text{min}}$ values of conventional shortened Hamming and BCH codes with the theoretical upper bound values of $d_{\text{min}}$ obtained from [27][29][29]. The number of nullified information symbol positions is given by $s$. It is observed that the $d_{\text{min}}$ values of these shortened codes are not optimal, in the sense that they do not meet the theoretical upper bound values. For example, when a Hamming(15, 11) code is shortened by 7 information symbols using the conventional shortening scheme as described in Section 2.3.2, the shortened Hamming(8,4) code has a $d_{\text{min}}$ of 3 while the upper bound states that the best possible (8,4) code is able to have a $d_{\text{min}}$ value of 4. This observation
motivates us to investigate other ways of code shortening with a view to maximize the \( d_{\text{min}} \) value or minimize the corresponding \( A_{d_{\text{min}}} \) value of the shortened codes.

Table 4.1. Comparison of \( d_{\text{min}} \) of conventional shortened codes and theoretical upper-bound values of \( d_{\text{min}} \)

<table>
<thead>
<tr>
<th>Code Type</th>
<th>((n, k)) of Original Code</th>
<th>(s)</th>
<th>(d_{\text{min}}) of (d_{\text{min}}) Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamming</td>
<td>(15, 11)</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(31, 26)</td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(63, 57)</td>
<td>31</td>
<td>3</td>
</tr>
<tr>
<td>BCH</td>
<td>(15, 7)</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(15, 5)</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>(31, 21)</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(31, 11)</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>(31, 6)</td>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

4.2 Ordered Shortening

4.2.1 Shortening from the Parity Check Matrix Point of View

A systematic codeword \( e \) has the form \( e = [u, m] \) where \( u = [u_1, u_2, \ldots, u_{n-k}] \) and \( m = [m_1, m_2, \ldots, m_k] \) constitute the parity-check portion and information message of the codeword respectively. Given the parity check matrix \([H] = [I_{n-k} P^T]\) where \([I_{n-k}] = [i_1, i_2, \ldots, i_{n-k}]\),
... \cdot \mathbf{i}_{n-k} \) and \( \mathbf{[P^T]} = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n, \ldots, \mathbf{p}_k] \), the codeword \( \mathbf{e} \) of a code satisfies the condition \( \mathbf{e}[\mathbf{H}^T] = \mathbf{0} \) (see Section 2.1.3). That is,

\[
\mathbf{u}_1 \mathbf{i}_1 + \mathbf{u}_2 \mathbf{i}_2 + \ldots + \mathbf{u}_{n-k} \mathbf{i}_{n-k} + \mathbf{m}_1 \mathbf{p}_1 + \mathbf{m}_2 \mathbf{p}_2 + \ldots + \mathbf{m}_p \mathbf{p}_i + \ldots + \mathbf{m}_k \mathbf{p}_k = \mathbf{0}. \tag{4.1}
\]

When a particular information symbol \( \mathbf{m}_i \) is shortened, its value is permanently put to zero (nullified), hence its contribution \( \mathbf{m}_i \mathbf{p}_i \) in (4.1) will be removed. This is equivalent to taking away the corresponding column \( \mathbf{p}_i \) from the matrix \( \mathbf{[P^T]} \) or \( \mathbf{[H]} \). Hence, there is a one-to-one correspondence between the respective \( \mathbf{m}_i \) and \( \mathbf{p}_i \) as shown by the arrow in Fig. (4.1).

4.2.2 Principle of Ordered Shortening

In order to obtain shortened codes with large \( d_{min} \) value, we propose to shorten a code by removing codewords with the smallest Hamming distances first. As explained in Section 2.1.3, the minimum distance of a systematic linear block code is equal to the smallest number of columns from \( \mathbf{[H]} = [\mathbf{I}_{n-k} \mathbf{P}^T] \) that sum to \( \mathbf{0} \). Since \( \mathbf{[I}_{n-k} \) is an identity matrix, any number of its columns will not sum to \( \mathbf{0} \). Given that \( \mathbf{[I}_{n-k} = [i_1, i_2, \ldots, i_{n-k}] \), the set \( \{i_{basis}\} = \{i_1, i_2, \ldots, i_{n-k}\} \) forms the basis for the column-space of the parity check.
matrix \([H] = [I_{n-k} \mathbf{P}^T]\), which implies that any column (or sum of columns) in \([\mathbf{P}^T]\) = \([p_1, p_2, ..., p_n, ..., p_k]\) can be summed with the basis vectors from \(\{I_{basis}\}\) to give the zero vector \(0\).

Suppose we select certain \(n_c\) columns from \([\mathbf{P}^T]\). Define the indicator vector \(z = [z_1, z_2, ..., z_i, ..., z_k]\) such that

\[
z_i = \begin{cases} 
1, & \text{if } p_j \text{ is selected} \\
0, & \text{if } p_j \text{ is not selected}
\end{cases}
\] (4.2)

For example, \(z = [1 \ 1 \ 0 \ 0]\) corresponds to the selection of \(p_1\) and \(p_2\). By selecting these \(n_c\) columns from \([\mathbf{P}^T]\), we are actually selecting a particular systematic codeword \(e\) with the information message \(m = z\). That is, the selection of a codeword with information weight \(n_c\). The information weight is defined as the number of ones in the information message \(m\) of a codeword.

Let \(v\) be the vector formed by the modulo-2 summation of these selected \(n_c\) distinct column vectors \(p_i\). Since \(v\) can be summed with the basis vectors from \(\{I_{basis}\}\) to give the zero vector \(0\), the weight \(w_v\) of \(v\) will then indicate the number of basis vectors needed to give the zero vector \(0\). \(w_v\) is also the parity check weight of the codeword i.e. the number of ones in the parity check sequence of the codeword. The Hamming distance \(d\) of the codeword with information message \(m = z\) is hence given by \(d = n_c + w_v\).

If \(d_{min}\) is the minimum Hamming distance of the original \((n, k)\) code, the codewords with weight \(d_{min}\) will hence give a weight of \(w_v = d_{min} - n_c\) for \(v\), the sum of \(n_c\) columns from \([\mathbf{P}^T]\). This property enables us to identify the codewords with weight \(d_{min}\) to remove by shortening the associated information symbol positions. This process is explained in the algorithm below, followed by an example.
4.3 Algorithm

Given that \([P_1, P_2, \ldots, P_n, \ldots, P_k]\), the set \(\{v\}_n\) is the set of \(v\) formed by all the possible modulo-2 summation of \(n_c\) distinct \(p_i\) identified by \(z\) (4.2), i.e.

\[
\begin{align*}
\text{if } n_c = 1, & \quad \{v\}_n = \{p_i\}_{\text{all } i} \\
\text{if } n_c = 2, & \quad \{v\}_n = \{p_i \oplus p_j\}_{\text{all } i \neq j} \\
\text{if } n_c = 3, & \quad \{v\}_n = \{p_i \oplus p_j \oplus p_k\}_{\text{all } i \neq j \neq k} \\
& \vdots
\end{align*}
\]

(4.3)

for \(1 \leq i, j, k \leq k\) and \(1 \leq n_c \leq d_{\text{min}}\).

The steps of the ordered shortening scheme are:

1. Starting with \(n_c = 1\), form all possible \(v\) vectors using (4.3).

2. The weight \(w_n\) of each \(v\) is computed. For example, if \(v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\), its weight \(w_n = 2\).

3. Pick all \(z\) with \(w_n = d_{\text{min}} - n_c\). They correspond to the set of minimum-Hamming-distance codewords denoted by \(\{z\}_n\).

4. Record the number of times that each \(z_i = 1\) has occurred in the set of \(\{z\}_n\) from step 3.

5. Shorten (nullify) the information symbol position corresponding to the highest frequency of \(z_i = 1\) occurrence.

6. Remove those \(z\) vectors identified in step 5 from \(\{z\}_n\).

7. Repeat steps 5 and 6 till all the \(z\) vectors in \(\{z\}_n\) are removed, or until the number of shortened (or nullified) positions \(s\) has been met, whichever occurs earlier.
Chapter 4 – Ordered Shortening

8. Increment \( n_c \) and repeat step 1-7 until \( n_c = d_{\text{min}} \).

Note that if \( n_c > d_{\text{min}} \), all the minimum Hamming distance codewords have already been removed.

4.3.1 Example of Ordered Shortening

We shall now illustrate the above shortening algorithm with a simple example. A small code, Hamming(15, 11) with \( d_{\text{min}} = 3 \), is used in this example. Its parity check matrix is given by

\[
[H] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

With \( n_c = 1 \), \( \{v\}_{n_c=1} = \{p_1, p_2, \ldots, p_{11}\} \). The set of information sequences \( \{z\}_{n_c} \) with information weight \( n_c = 1 \) from the systematic codewords are

\[
\{z\}_{n_c=1} = \begin{cases}
z_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
z_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
z_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
z_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
z_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
z_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
z_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \\
z_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \\
z_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \\
z_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
z_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{cases}
\]

(4.5)
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For ease of identification, we have labeled these information sequences from $z_i$ to $z_{11}$. The parity-check weights of the codewords identified by $\{z\}_n$ are given by the weights of the respective columns from $\left[ P^T \right]_n$ to be 2, 2, 2, 2, 3, 3, 4, 3, and 2. The $z$ patterns from (4.5) that satisfy the criterion $w_n = d_{\min} - n_c = 2$ are

$$\{ z | w_n = 2 \}_{n_c=1} = \begin{cases} z_1 = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], \\ z_2 = [0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], \\ z_3 = [0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], \\ z_4 = [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], \\ z_5 = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], \\ z_6 = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0], \\ z_7 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \end{cases}.$$  \hspace{1cm} (4.6)

These are the information sequences of the identified minimum Hamming distance codewords. Hence if the desired $s = 6$, $z_1$, $z_2$, $z_3$, $z_5$, $z_6$ and $z_{11}$ are to be removed from $\{z\}_n$. Therefore, information symbol positions 1, 2, 3, 5, 6 and 11 are to be shortened and the parity check columns $p_1$, $p_2$, $p_3$, $p_5$, $p_6$ and $p_{11}$ will be removed from the parity check matrix. If $s = 7$, we have to increment $n_c$ and repeat the above using the shortened parity check matrix $\left[ H_r \right] = \left[ I_4 \ p_4 \ p_7 \ p_8 \ p_9 \ p_{10} \right]$ which consists of the original identity matrix $I_4$ and the remaining $p_n$.

With $n_c = 2$, the $z$-patterns that meet the criterion $w_n = d_{\min} - n_c = 1$ are

$$\{ z | w_n = 1 \}_{n_c=2} = \begin{cases} z_1 = [0 \ 0 \ 1 \ 1 \ 0], \\ z_2 = [0 \ 1 \ 0 \ 1 \ 0], \\ z_3 = [1 \ 0 \ 0 \ 1 \ 0] \end{cases}.$$  \hspace{1cm} (4.7)

with the columns corresponding to $p_4$, $p_7$, $p_8$, $p_9$ and $p_{10}$ respectively. For convenience, we
have labeled them as $z_i$ to $z_j$. These are the information sequences of minimum Hamming distance codewords with $n_c=2$. We may remove these minimum Hamming distance codewords by shortening the corresponding information bit 4, 7, 8, 9 or 10. Clearly the information bit to shorten is the one associated with the most number of minimum Hamming distance codewords i.e. bit 9 in this case. Combined with the previous results, the information symbol positions to be shortened for $s=7$ are therefore 1, 2, 3, 5, 6, 11 and 9.

In the above steps are repeated for $n_c = d_{\text{min}} = 3$, no $z$-patterns with $w_{n_c} = d_{\text{min}} - n_c = 0$ are found. Hence, performing ordered shortening with $s=7$ on a Hamming$(15, 11)$ results in a shortened code with increased $d_{\text{min}}=4$.

Note that the search space of the proposed ordered shortening algorithm is not more than $\sum_{n_c=1}^{d_{\text{min}}} \binom{k}{n_c}$ or $\sum_{n_c=1}^{k} \binom{k}{n_c}$ whichever is smaller. Hence the algorithm is generally much more efficient than a brute-force search over all combinations of $k$ information bits.
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4.4 Results

4.4.1 Ordered-Shortened Hamming and BCH Codes

Table 4.2 shows the set of information symbol positions \( \{i_s\} \) to be shortened until an increase in \( d_{\text{min}} \) is achieved for Hamming and BCH codes. In the table, the symbols \( n, k \) and \( d_{\text{min}} \) are parameters of the original un-shortened code. The increased minimum Hamming distance value is \( d_{\text{min}} + 1 \). From the table, we can see that ordered-shortened codes reach an increase in \( d_{\text{min}} \) faster than the conventional-shortened codes. Moreover, the conventional shortening scheme does not produce an increase in \( d_{\text{min}} \) at all for some codes such as Hamming(15, 11), Hamming(31, 26), BCH(15, 7), BCH(15, 5) and BCH(31, 11).
Table 4.2. Information symbol positions to shorten to achieve an increased $d_{\text{min}}$ for Hamming and BCH codes with $n = 15$ and 31

Table 4.2a. Information symbol positions to shorten to achieve an increased $d_{\text{min}}$ for Hamming codes with $n = 15$ and 31

<table>
<thead>
<tr>
<th>Hamming</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Original code</strong></td>
<td>${i,}$</td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td>$k$</td>
<td>$d_{\text{min}}$</td>
</tr>
<tr>
<td>15</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>31</td>
<td>26</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4.2b. Information symbol positions to shorten to achieve an increased $d_{\text{min}}$ for BCH codes with $n = 15$ and 31

<table>
<thead>
<tr>
<th>BCH</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Original code</strong></td>
<td>${i,}$</td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td>$k$</td>
<td>$d_{\text{min}}$</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>${1, 2, 5}$</td>
</tr>
<tr>
<td>21</td>
<td>5</td>
<td>${4, 5, 14, 15, 16, 1, 8, 10, 19, 20, 21}$</td>
</tr>
<tr>
<td>31</td>
<td>16</td>
<td>7</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>${1, 2, 10, 11, 4}$</td>
</tr>
</tbody>
</table>

*No increase in $d_{\text{min}}$
4.4.2 Comparison of Minimum Hamming Distance with Theoretical Upper Bound

Fig. 4.2 shows the minimum Hamming distance $d_{\text{min}}$ of Hamming(31, 26, 3) shortened by the conventional shortening and the proposed ordered shortening schemes, alongside the corresponding theoretical upper bound values of $d_{\text{min}}$ obtained from [27][29][29] for the best possible (31-s, 26-s) code. It is clear that the ordered shortening scheme produces an increase in $d_{\text{min}}$ of 1 after shortening 15 information symbols (i.e. $s = 15$), while the conventional shortening scheme does not, even after shortening all information symbols. Furthermore, the $d_{\text{min}}$ of ordered-shortened code meets the theoretical upper bound.

Fig. 4.2. Comparison of $d_{\text{min}}$ between conventional shortening, ordered shortening and the best (31-s, 26-s) possible code. Original code= Hamming(31,26,3).
4.4.3 Comparison of Minimum Weight Enumerator

Next we study the effect of ordered shortening on the weight enumerator $A_d$ before an increase in $d_{\text{min}}$ is achieved. Fig. 4.3a and 4.3b show the weight enumerator $A_d$ of s-Hamming(18, 13) and s-PC(18, 13)$^2$ respectively, obtained from Hamming(31, 26) with $s = 13$. Note that only the smallest few (hence most significant) values of $d$ are considered here. According to [9], the smallest three values of $d$ of a product code made up of non-extended Hamming codes are 9, 12 and 15, and the corresponding weight enumerators are

$$A_d = \begin{cases} A_3^h, & \text{if } d = 9 \\ A_4^h A_4^v A_3^h + A_5^h A_3^v, & \text{if } d = 12 \\ A_5^h A_5^v A_3^h, & \text{if } d = 15 \end{cases} \quad (4.8)$$

where the superscripts $h$ and $v$ refer to the horizontal and vertical component codes of the product code respectively. Hence, the weight enumerator values $A_9$, $A_{12}$ and $A_{15}$ of Fig. 4.3b are computed using (4.8) based on the $A_3$, $A_4$ and $A_5$ values from Fig. 4.3a.
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Fig. 4.3. Comparison of $A_d$ between ordered and conventional shortening for (a) $s$-Hamming(18, 13) and (b) $s$-PC(18, 13).

This shows that the proposed ordered shortening scheme produces $s$-Hamming(18, 13) with smaller $A_d$ values than conventional shortening for $d = 3$ and 5. This is also true for $s$-PC(18, 13) as shown in Fig. 4.3b, for all values of $d = 9, 12$ and 15. This shows that besides producing increased $d_{mm}$ faster (with smaller $A_d$), the ordered shortening scheme is also able to produce smaller $A_d$ values for the smaller $d$ terms. Note also that although the $A_d$ value of $s$-PC(18, 13) is contributed by both the $A_3$ and $A_4$ values of $s$-Hamming(18, 13), Fig. 4.3a shows that the proposed ordered shortening scheme produces $s$-Hamming(18, 13) with smaller $A_d$ values than conventional shortening for $d = 3$ and 5.

Since the proposed ordered shortening scheme is capable of producing larger $d_{mm}$ or smaller $A_d$ values, the BER performance of ordered shortened code is expected to be...
better than that of the conventional-shortened code. We shall verify this in the next section.

4.4.4 Comparison of Simulated BER

Fig. 4.4 shows the simulated BER and the asymptotic BER bound (obtained from (2.7) using only the $d_{\text{min}}$ term) for s-PC(17,12)$^3$ with ordered and conventional shortening in AWGN channel. The simulation results are obtained using Pyndiah’s algorithm with 6 test positions and 4 iterations. Both the bound and simulated results show that the ordered-shortened code has lower BER than the conventional-shortened code. Specifically, at a BER of $10^{-6}$, the ordered-shortened code outperforms the conventional-shortened code by about 0.8 dB.

Fig. 4.5 shows the simulated BER and the asymptotic BER bound for s-PC (33,27)$^3$ with ordered and conventional shortening in AWGN channel. At a BER of $10^{-6}$, the ordered-shortened code’s outperforms the conventional-shortened code by about 0.6 dB.
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Fig. 4.4. Simulated and analytical BER performance of s-PC(17, 12) with ordered and conventional shortening in AWGN channel

Fig. 4.5. Simulated and analytical BER performance of s-PC(33, 27) with ordered and conventional shortening in AWGN channel
Fig. 4.6 shows the simulated BER and asymptotic BER bound (obtained from (2.9) using only the $d_{\text{min}} = 9$ term) for s-PC $(17, 12)^2$ with ordered and conventional shortening in Rayleigh fading channel. Better BER performance is again observed for the ordered-shortened code. At BER of $10^{-6}$, the difference is about 1 dB.

![Graph showing BER performance](image)

Fig. 4.6. Simulated and analytical BER performance of s-PC$(17, 12)^2$ with ordered and conventional shortening in Rayleigh fading channel.

Note that for Fig. 4.4, 4.5 and 4.6, the $d_{\text{min}} (=9)$ of the conventional and ordered shortened codes are the same. The performance gain is hence mainly due to a reduced $A_{\eta_{\text{min}}}$. 

Fig. 4.7 shows the simulated BER and the asymptotic BER bound for s-PC$(20, 10)^2$ with ordered and conventional shortening in AWGN channel. It illustrates the case...
where the $d_{\text{min}}$ of the ordered-shortened codes is larger than the $d_{\text{min}}$ of the conventional-
shortened codes for the same amount of shortening ($s = 11$). The conventional-shortened code has $d_{\text{min}} = 5 \times 5 = 25$ while the ordered-shortened code has $d_{\text{min}} = 6 \times 6 = 36$. The bigger $d_{\text{min}}$ has evidently resulted in a steeper BER slope for the ordered-shortened code and hence a diverging BER gap compared to the conventional shortened code. This makes the ordered-shortened product code particularly attractive for systems requiring very low BER.

![Graph showing BER performance](image)

Fig. 4.7. Simulated and analytical BER performance of s-PC(20, 10) with ordered and conventional shortening in AWGN channel
Chapter 5

Turbo Decoding of Product Code in Pulsed Jamming Channels

In this chapter, we extend Pyndiah’s turbo decoding algorithm for product codes (PC) to the pulsed jamming channels and benchmark its performance with the BER bounds.

5.1 Pulsed Jamming Noise Model

We consider an interferer who jams a fraction $\rho$ ($0 < \rho < 1$) of the symbols in a codeword. The fraction of coded symbols not jammed is equal to $(1 - \rho)$. $\rho$ is also known as the duty cycle of the jammer. We model the jamming signal as white Gaussian noise whose one-sided power spectral density (PSD) is $N_j / \rho$ when the jammer is on, and 0 when the jammer is off [30]. The average jammer’s PSD (one-sided) is then $N_j$.

Hence, the one-sided PSD of the jamming noise is
Chapter 5 – Turbo Decoding of Product Code in Pulsed Jamming Channels

\[ N_j = \begin{cases} \frac{N_j}{\rho}, & \text{with probability } \rho \\ 0, & \text{with probability } (1-\rho) \end{cases} \]  

(5.1)

and the one-sided PSD of the total noise \( N_T \) for a particular symbol is then given by

\[ N_T = \begin{cases} \frac{N_j + N_0}{\rho}, & \text{if jammer is on} \\ N_0, & \text{if jammer is off} \end{cases} \]  

(5.2)

where \( N_0 \) is the one-sided gaussian noise spectral density.

We further assume that the probability that a symbol is jammed is independent of the rest of the symbols, i.e. the jamming effect is memoryless across the coded symbols.

### 5.2 BER Bounds for Pulsed Jamming Channels

In this section, we shall derive the BER union bounds for a pulsed jamming channel for use as reference for the simulated BER results to be obtained later on. In the derivation, we assume that the decoder has the knowledge of the jamming state information (JSI) which includes \( N_j \) and \( \rho \), and the channel state information (CSI) which includes the fading coefficients and \( N_0 \).

#### 5.2.1 Pulsed Jamming AWGN Channel with JSI

The upper bound on the BER of a linear code in AWGN channel is shown in (2.7). To find the corresponding BER bound for a pulsed jamming channel with AWGN channel noise, we need to first condition the pairwise error probability (2.8) on the
number of jammed symbols \( n_j \) out of \( d \) symbols in a codeword and then average this conditioned pairwise error probability over the distribution of \( n_j \).

Based on the assumption that the probability that a symbol is jammed is memoryless, we can model \( n_j \) as having a binomial distribution with probability of success, \( p \) (probability of a symbol being jammed) and the maximum value of \( n_j \), \( d \) (code Hamming distance).

Therefore the pdf of \( n_j \) is given by

\[
P(n_j) = \sum_{n_j=0}^{d} \binom{d}{n_j} p^{n_j} (1-p)^{d-n_j}
\]

(5.3)

The pairwise error probability in a pulsed jamming AWGN channel is then given by

\[
P_e(d, r_c) = P_h(d, r_c | n_j).P(n_j)
\]

\[
= \sum_{n_j=0}^{d} \binom{d}{n_j} p^{n_j} (1-p)^{d-n_j}.P_d(d, r_c | n_j)
\]

(5.4)

where \( P_d(d, r_c | n_j) \) is the pairwise error probability conditional on \( n_j \) of the \( d \) symbols being jammed.

Given that \( n_j \) out of \( d \) symbols in a codeword are jammed, we can separate these \( d \) symbols into \( n_j \) jammed and \( (d - n_j) \) unjammed symbols with different noise PSD as given in (5.2) respectively and modify (2.8) accordingly to give

\[
P_e(d, r_c | n_j) = Q \left( \sqrt{2r_cE_p \left( \frac{d-n_j}{N_0} + \frac{n_j}{N_0+N_J/p} \right)} \right)
\]

(5.5)
Using (5.4) in (2.7), the soft decoding bit error probability $P_b$ in a pulsed jamming AWGN channel is then upper-bounded by

$$P_b \leq \sum_{d}^{d} \frac{d}{n} A_d \cdot P_d \left( d, r | n_j \right) P \left( n_j \right)$$

where $P_d \left( d, r | n_j \right)$ is given in (5.5).

### 5.2.2 Pulsed Jamming Rayleigh Fading Channel with JSI and CSI

The approach to finding the BER union bound for a pulsed jamming Rayleigh fading channel with JSI and CSI is similar to that in Section (5.2.1), except that we have to condition the pairwise error probability $P_d \left( d, r | a, n_j \right)$ on 2 random variables now:

1. $a = [a_1, \ldots, a_d]$, the fading coefficient vector for $r = [r_1, \ldots, r_d]$ and,
2. $n_j$, the number of jammed symbols out of $d$ symbols.

Subsequently, we will also need to average $P_d \left( d, r | a, n_j \right)$ over the pdfs of $a$ and $n_j$.

Following the derivation of (5.5), we separate the $d$ symbols from (2.10) into jammed and unjammed symbols where the $n_j$ jammed and $(d - n_j)$ unjammed symbols have noise PSD as given in (5.2) to obtain

$$P_d \left( d, r | n_j, a \right) = Q \left( \sqrt{2r \left( \frac{E_b}{N_f / \rho + N_0} \sum_{m=1}^{n_j} a_m^2 + \frac{E_b}{N_0} \sum_{i \neq n_j+1} a_i^2 \right)} \right)$$

Noting (2.15), $P_d \left( d, r | a, n_j \right)$ can be expressed as
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\[ P_d(d, r_c | n_j, a) = \frac{1}{\pi} \int_0^{\pi/2} e^{-\left( \frac{r_e E_b}{N_j / \rho + N_0} \sum_{\text{in}} \theta_i + \frac{r_e E_b}{N_0} \sum_{\eta=1}^{N} d_{\eta} \right)} \sin^2 \phi \ d \phi \]  

(5.8)

To compute the average error probability \( \overline{P}_d(d, r_c | n_j) \), \( P_d(d, r_c | n_j, a) \) is averaged over \( a \) to give

\[
\overline{P}_d(d, r_c | n_j) = \int_a p_a(a) P_d(d, r_c | n_j, a) \ da
\]

\[
= \int_{a_1}^{a_2} \cdots \int_{a_d}^{a_{d+1}} P_d(d, r_c | n_j, (a_1, a_2, \ldots, a_d)) \cdot p_a(a_1, a_2, \ldots, a_d) \ da_1 \cdots \ da_d
\]

(5.9)

where \( p_a(a_1, a_2, \ldots, a_d) \) is given in (2.13). Assuming independent Rayleigh fading, the \( d \)-dimensional integral in (5.9) reduces to a product of integrals over each \( a_i \).

Substituting (5.8) and (2.13) into (5.9), it can be shown that (see Appendix C for the derivation)

\[
\overline{P}_d(d, r_c | n_j) = \frac{1}{\pi} \int_0^{\pi/2} \left( \frac{\sin^2 \phi}{\left( \frac{r_e E_b}{N_j / \rho + N_0} + \sin^2 \phi \right)^{n_j}} \right)^{d-n_j} \left( \frac{\sin^2 \phi}{\left( \frac{r_e E_b}{N_0} + \sin^2 \phi \right)} \right)^{d-n_j} \ d \phi
\]

(5.10)

The soft decoding bit error probability \( P_b \) in a pulsed jamming Rayleigh fading channel is then upper-bounded by

\[
P_b \leq \sum_d \sum_n A_d \overline{P}_d(d, r_c | n_j) \cdot P(n_j)
\]

\[
\leq \sum_d \sum_n A_d \sum_{n_j=0}^{d} \left( \frac{d}{n_j} \right)^{n_j} (1-\rho)^{d-n_j} \cdot \overline{P}_d(d, r_c | n_j)
\]

(5.11)
5.3 Pyndiah's SISO Decoder for Pulsed Jamming Channel

5.3.1 Decoding in Pulsed Jamming AWGN Channel

5.3.1.1 Case I: Decoding with JSI

In addition to the jammer's PSD $N_j$ and jamming factor $\rho$, jammer state information (JSI) includes the knowledge of the number of jammed symbols and in particular, which symbols are jammed. In practical situations, estimation of the JSI is required to predict whether a symbol has been jammed. This, however, will not be discussed in this thesis.

We will mainly derive 2 new results:
- Channel reliability values with JSI
- Decoder soft output with JSI

Channel Reliability Values:

The channel reliability value generated by the demodulator from $r$ and jammer state information $b = [b_1, \ldots, b_j, \ldots, b_n]$ using the log-likelihood ratio (LLR) of decision $r_j$ is given by

$$\Lambda(r_j) = \ln \left( \frac{\Pr \left\{ x_j = +1 \mid r_j, b_j \right\}}{\Pr \left\{ x_j = -1 \mid r_j, b_j \right\}} \right)$$

(5.12)

where $b_j$ is defined as

$$b_j = \begin{cases} 0, & \text{if bit } j \text{ is unjammed} \\ 1, & \text{if bit } j \text{ is jammed} \end{cases}$$

(5.13)
With the total PSD given in (5.2), the pdf of each pulse-jammed-AWGN-channel noise sample $n_j$ is given by

\[
p(n_j | b_j) = \begin{cases} 
\frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left[ -\frac{n_j^2}{2\sigma_n^2} \right], & \text{if } b_j = 0 \\
\frac{1}{\sqrt{2\pi \left( \sigma_n^2 + \frac{\sigma_j^2}{\rho} \right)}} \exp \left[ -\frac{n_j^2}{2 \left( \sigma_n^2 + \frac{\sigma_j^2}{\rho} \right)} \right], & \text{if } b_j = 1
\end{cases}
\]  
(5.14)

where $\sigma_n^2$ is the AWGN noise’s variance and $\frac{\sigma_j^2}{\rho}$ is the jammer noise’s variance.

Given that $x_j$ is transmitted, the pdf of a received symbol $r_j$ from a pulsed jamming AWGN channel is

\[
p(r_j | x_j, b_j) = \begin{cases} 
\frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left[ -\frac{(r_j - x_j)^2}{2\sigma_n^2} \right], & \text{if } b_j = 0 \\
\frac{1}{\sqrt{2\pi \left( \sigma_n^2 + \frac{\sigma_j^2}{\rho} \right)}} \exp \left[ -\frac{(r_j - x_j)^2}{2 \left( \sigma_n^2 + \frac{\sigma_j^2}{\rho} \right)} \right], & \text{if } b_j = 1
\end{cases}
\]  
(5.15)

Using Bayes’ rule, the reliability value in (5.12) can be shown to be

\[
\Lambda(r_j) = \begin{cases} 
\frac{2}{\sigma_n^2} r_j, & \text{if } b_j = 0 \\
\frac{2}{\sigma_n^2 + \frac{\sigma_j^2}{\rho}} r_j, & \text{if } b_j = 1
\end{cases}
\]  
(5.16)

The absolute value of this channel reliability value, $|\Lambda(r_j)|$, is used to identify the least reliable received bits for forming the test patterns in the Pyndiah/Chase algorithm (see Section 2.5.1.3).
Chapter 5 – Turbo Decoding of Product Code in Pulsed Jamming Channels

Soft Output from Decoder:

The reliability of decision $d_j$ at the output of the channel decoder defined using log-likelihood ratio (LLR) is given by

$$\Lambda(d_j) = \ln \left( \frac{\sum_{c' \in \mathcal{C}^t_j} p(r | x = c', n_j)}{\sum_{c' \in \mathcal{C}^t_j} p(r | x = c', n_j)} \right).$$

(5.17)

It is the same as (2.26) except that now we have conditioned it upon the knowledge of the symbols being jammed $b$ and hence the number of jammed symbols $n_j$, out of the received $n$-tuple vector $r$.

Assuming that the received symbols are independent, $p(r | x = c', n_j)$ can be expressed as

$$p(r | x = c', n_j) = \prod_{j=1}^{n} p(r_j | x_j = c_j, b_j)$$

(5.18)

which can be shown to be

$$p(r | x = c', n_j) = \left[ \frac{1}{\sqrt{2\pi(\sigma_n^2 + \sigma_j^2/\rho)}} \right]^{n_j} \cdot \frac{1}{\sqrt{2\pi\sigma_n^2}} \cdot \exp \left[ -\sum_{l=1}^{n_j} \frac{(r_l - c_l)^2}{2(\sigma_n^2 + \sigma_j^2/\rho)} - \sum_{l=1, l \neq j}^{n} \frac{(r_l - c_l)^2}{2\sigma_n^2} \right].$$

(5.19)

If the received symbols within a codeword are not independent of one another, it implies that there is correlation between the fade coefficients of the received symbols. Hence
(5.18) would not be a simple pure product of the individual pdf of the individual symbol. Decoding performance is expected to be worse due to inter-symbol correlation.

We have separated the \( n \) symbols of a codeword into \( n_J \) jammed and \( (n - n_J) \) unjammed symbols with noise PSD as given in (5.2).

Using (5.19) in (5.17),

\[
\Lambda(d_J) = \ln \left[ \sum_{e^c \in \{s_{j}^{+1}\}} \exp \left( \sum_{l=1}^{n_J} \frac{(r_l - c_l)^2}{\sigma_n^2 + \sigma_j^2 / \rho} - \sum_{l=n_J+1}^{n} \frac{(r_l - c_l)^2}{2\sigma_n^2} \right) \right].
\]

Following Section 2.5.2.1, the two codewords \( c^{+1(j)} \) and \( c^{-1(j)} \) are also considered here to simplify the computation. Hence (5.20) can be shown to be (See Appendix D for the derivation)

\[
\Lambda(d_J) = \sum_{l=1}^{n_J} \frac{(r_l - c_l^{+1(j)})^2 - (r_l - c_l^{-1(j)})^2}{2\sigma_n^2 + \sigma_j^2 / \rho} + \sum_{l=n_J+1}^{n} \frac{(r_l - c_l^{-1(j)})^2 - (r_l - c_l^{+1(j)})^2}{2\sigma_n^2} + \ln \left( \frac{\sum P_i}{\sum Q_i} \right)
\]

where

\[
P_i = \exp \left( \sum_{l=1}^{n_J} \frac{(r_l - c_l^{+1(j)})^2 - (r_l - c_l^{-1(j)})^2}{2\sigma_n^2 + \sigma_j^2 / \rho} + \sum_{l=n_J+1}^{n} \frac{(r_l - c_l^{-1(j)})^2 - (r_l - c_l^{+1(j)})^2}{2\sigma_n^2} \right), \text{ with } c' \in \{s_{j}^{+1}\}
\]

and
\[ Q = \exp \left( \frac{\sum_{i=1}^{n_j} (r_i - c_i^{(j)})^2 - (r_i - c_i^1)^2}{2\left(\sigma_n^2 + \frac{\sigma_j^2}{\rho}\right)} + \frac{\sum_{i\in\pi_j+1} (r_i - c_i^{(j)})^2 - (r_i - c_i^2)^2}{2\sigma_n^2} \right), \quad \text{with } c^i \in \{s_i^1, s_i^2\}. \] (5.23)

At high SJR and high SNR, \( \sigma_j \to 0 \) and \( \sigma_n \to 0 \), \( \sum_i P_i \approx \sum_i Q_i \Rightarrow \sum_i P_i / \sum_i Q_i \to 1 \), thus the third term in (5.21) tends to 0.

An approximation of \( \Lambda(d_j) \) can then be obtained as

\[ \Lambda(d_j) \approx \Lambda'(d_j) = \frac{\sum_{i=1}^{n_j} (r_i - c_i^{(j)})^2 - (r_i - c_i^1)^2}{2\left(\sigma_n^2 + \frac{\sigma_j^2}{\rho}\right)} + \frac{\sum_{i\in\pi_j+1} (r_i - c_i^{(j)})^2 - (r_i - c_i^2)^2}{2\sigma_n^2}. \] (5.24)

Define

\[ \sigma_i^2 = \begin{cases} \sigma_n^2, & \text{if unjammed} \\ \sigma_n^2 + \sigma_j^2 / \rho, & \text{if jammed} \end{cases} \] (5.25)

Therefore,

\[ \Lambda'(d_j) = \sum_{i=1}^{n_j} \frac{(r_i - c_i^{(j)})^2 - (r_i - c_i^1)^2}{2\sigma_i^2} \] (5.26)

which can be further simplified to

\[ \Lambda'(d_j) = \frac{2}{\sigma_j^2} r_j + \sum_{i=1}^{n_j} \frac{2r_j c_i^{(j)} p_i}{\sigma_i^3} \] (5.27)

where
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\[ p_i = \begin{cases} 
0, & \text{if } c_i^{(t)} = c_i^{(t)} \\
1, & \text{if } c_i^{(t)} \neq c_i^{(t)}. 
\end{cases} \]

It can be seen that (5.27) is consistent with the SISO decoding model in [31] where the 1st term is the channel LLR values (as obtained earlier) and the 2nd term is the extrinsic information gleaned from the decoding process.

We can see that in a jammed AWGN channel, we cannot normalize \( \Lambda'(d_j) \) by the noise variance since it is no longer a constant over a codeword. However, we can still normalize (5.27) by the constant 2. As such, we obtain

\[ r'(d_j) = \frac{\Lambda'(d_j)}{2} = \frac{1}{\sigma_j} r_j + \sum_{i=1}^{n} \frac{r_i c_i^{(t)} p_i}{\sigma_j^2} \]

which is equivalent to

\[ r'(d_j) = \sum_{i=1}^{n} \left( \frac{r_i - c_i}{4\sigma_i^2} \right)^2. \]

To be consistent with the notations used in (2.37), we express the soft output \( r'(d_j) \) from the decoder as

\[ r'(d_j) = \left( \frac{\sum_{i=1}^{n} \left( \frac{r_i - c_i}{4\sigma_i^2} \right)^2}{\sigma_i^2} \right) d_j \]

where \( c = [c_1, c_2, \ldots, c_n] \) and \( d = [d_1, d_2, \ldots, d_n] \) have the same definitions as given in Section 2.5.2.2.

To be consistent with the form given in (5.28), for cases where there is no competing codeword, we propose to use the reliability factor \( \beta \) such that it is normalized by the symbol's noise variance \( \sigma_i^2 \). Thus,
where $\beta$ can be obtained from (2.42).

The extrinsic information can therefore be expressed as

$$w_j = r'_j - \frac{r_j}{\sigma_j^2} = \begin{cases} \left( \sum_{i=1}^{t} \left( \frac{(r_i - c_i)^2 - (r_i - d_i)^2}{4\sigma_i^2} \right) d_j - \frac{r_j}{\sigma_j^2} \right) & \text{if } c \text{ exists} \\ \frac{\beta}{\sigma_j^2} d_j - \frac{r_j}{\sigma_j^2} & \text{if } c \text{ does not exist} \end{cases}$$  \hspace{1cm} (5.32)

\subsection*{5.3.1.2 Case 2: Decoding without JSI}

Since there is no knowledge of JSI, the channel reliability values can be found from (2.19) and the soft output of the decoder is given by (2.37).

\subsection*{5.2.3 Decoding in Pulsed Jamming Rayleigh Fading Channel}

\subsection*{5.3.1.3 Case 1: Decoding with JSI and CSI}

Channel Reliability Values:

With $a_j$ being the fading coefficient of each received symbol $r_j$, the reliability value of each received symbol $r_j$ from the Pulsed Jamming Rayleigh fading channel is given by

$$\Lambda(r_j) = \begin{cases} \frac{2}{\sigma_n^2} a_j r_j, & \text{if } b_j = 0 \\ \frac{2}{\sigma_n^2 + \sigma_j^2 / \rho} a_j r_j, & \text{if } b_j = 1 \end{cases} \hspace{1cm} (5.33)$$
Soft Output from the channel Decoder:

Similarly, under the assumption that SJR and SNR are large, the soft output from the decoder \( r'(d_j) \) can be shown to be

\[
r'(d_j) = \frac{\sum_{i=1}^{n} \left( r_i - a_i c_i \right)^2}{4\sigma_i^2} - \left( \frac{\sum_{i=1}^{n} \left( r_i - a_i d_i \right)^2}{4\sigma_i^2} \right).
\]

(5.34)

Using the notations used in (2.37), the soft output \( r'(d_j) \) from the decoder is given by

\[
r'(d_j) = \left( \frac{\sum_{i=1}^{n} \left( r_i - a_i c_i \right)^2}{4\sigma_i^2} \right) d_j
\]

(5.35)

where \( c \) and \( d \) are have the same definitions as given in Section 2.5.2.2.

It is also noteworthy that the same soft output will result if the following is performed

\[
r'(d_j) = \left( \frac{\sum_{i=1}^{n} \left( a_i r_i - c_i \right)^2}{4\sigma_i^2} \right) d_j
\]

(5.36)

Similarly, for cases where there is no competing codeword, we propose to use the normalised reliability factor such that

\[
r'_j = \frac{\beta a_j d_j}{\sigma_i^2}
\]

(5.37)

where \( \beta \) can be obtained from (2.42).

5.3.1.4 Case 2: Decoding without JSI and CSI

With no knowledge of JSI and CSI, the channel reliability values can be found from (2.19) and the soft output of the decoder by (2.37).
5.4 Results and Discussions

We simulate the BERs for the following cases:

a) No utilization of JSI (and CSI for Rayleigh fading channel) in the decoding process.

b) Partial utilization of JSI (and CSI) in the decoding process, i.e. JSI (and CSI) are used to compute the extrinsic information, but not used to determine the test patterns (tp) for use in the Pyndiah's algorithm (see Section 2.5.1.3 for information on tp)

c) Full utilization of JSI (and CSI) in the decoding process to obtain the tp and to compute the extrinsic information.

Fig. 5.1 shows the asymptotic BER bound and its simulated values for PC(7, 4)² with \( \rho = 0.3 \) , \( E_b/N_0 = 7 \)dB and 4 iterations in a pulsed jamming AWGN Channel. The asymptotic BER bound is obtained using (5.6) using the \( d_{\text{min}} \) and \( A_{\text{min}} \) terms only and is hence a lower BER bound. The simulated BERs are obtained under cases a, b and c respectively. Under case a, (2.19) and (2.37) are used; under case b, (2.19) and (5.30) are used; under case c, (5.16) and (5.30) are used. In the simulations, the noise-variance weighted reliability factor from (5.31) and the \( \alpha \) weighting factor from (2.41) are used.
Fig. 5.1 shows that the asymptotic bound is a good estimate for the simulated BER with full JSI. As $E_b/N_j$ increases, the tightness of the bound also increases. At higher $E_b/N_j$ ($\geq 8$ dB), the full or partial use of JSI is not important as their simulated BER are about the same. However, at lower $E_b/N_j$, the simulated BER of partial JSI is higher than that of full JSI. At BER of $2 \times 10^{-4}$, the coding loss in term of $E_b/N_j$(dB) is about 1 dB. This can be explained using (5.16). As $E_b/N_j$ decreases, $\sigma_j^2$ becomes significant in (5.16). If this information is not used by the decoder (as in the partial JSI case), the least reliable bits due to jamming may not be correctly identified, the resultant test pattern will
hence be non-optimal. Fig. 5.1 also shows very poor BER results when JSI is not used (No JSI), as expected due to the large mismatch between the assumed and true noise PSD.

Fig. 5.2 shows the asymptotic BER bound and simulated BER values for PC(7, 4) in a pulsed jamming AWGN Channel with $E_b/N_0 = 7$dB and a smaller $\rho = 0.05$. The simulated BERs shown are for decoding with full, partial and no JSI. The bound is again a tight estimate for the simulated BER with full JSI. Furthermore, the simulated BER with full JSI is relatively flat i.e. there is no noticeable improvement to BER even as $E_b/N_0$ increases. This is because when $\rho$ is very small, the probability of a symbol being jammed is very small. Hence the BER approaches that of a AWGN channel without jamming.
Fig. 5.2. Asymptotic BER bound and simulated BER values for PC(7, 4) in Pulsed Jamming AWGN Channel with $\rho = 0.05$ and $E_b/N_0 = 7$dB

Fig. 5.3 shows the asymptotic BER bound and simulated BER values for PC(7, 4) in a pulsed jamming Rayleigh fading channel with $\rho = 0.3$, $E_b/N_0 = 7$dB and 4 iterations. The asymptotic BER bound is obtained using (5.11) using the $d_{\min}$ and $A_{d_{\min}}$ terms only. The simulated BERs are obtained under decoding with full, partial and no JSI respectively. Under no JSI, (2.19) and (2.37) are used; under partial JSI, (2.19) and (5.36) are used; under full JSI, (5.33) and (5.36) are used. In the simulations, the weighted reliability factor from (5.37) and $\alpha$ weighting factor from (2.41) are used.
Fig. 5.3. Asymptotic BER bound and simulated BER values for PC(7, 4) in Pulsed Jamming Rayleigh Fading Channel with $\rho = 0.3$ and $E_b/N_0 = 7$dB.

The result from Fig. 5.3 show that in pulsed jamming Rayleigh fading channel, decoding with partial or full use of JSI and CSI are both able to meet the lower BER bound, but not so if JSI and CSI are not used by the decoder.

Fig. 5.4 shows the asymptotic BER bound and simulated BER values for PC(7, 4) in a pulsed jamming Rayleigh fading channel with $E_b/N_0 = 7$dB, 4 iterations and a smaller $\rho = 0.05$. Similarly, the asymptotic bound is reached by decoding with full or partial JSI and CSI. Also, when full JSI and CSI are used, the simulated BER is relatively flat when $\rho$ is very small.
From the BER results obtained so far, we can see that both JSI and CSI are critical in obtaining correct test patterns and soft output in the decoding process of Pyndiah’s decoding algorithm. Also, the BER bounds (5.6) and (5.11) which we derive for pulsed jamming AWGN and pulsed jamming Rayleigh fading channel respectively are good estimates of the true BER.

Fig. 5.5 shows the BER lower bound and its simulated values for a larger product code PC(31, 26) in a pulsed jamming AWGN channel with $\rho = 0.3$, $E_b/N_0 = 7$dB and 4 iterations. Decoding with full JSI is used to obtain the simulated BER. Unlike Fig. 5.1 and 5.2, the original weighting factor $\alpha$ from (2.41) does not result in good BER in this
case. Hence, another set of manually optimized weighting factor given by $\alpha_{opt} = [0.0 \ 0.2 \\ 0.3 \ 0.5 \ 0.7 \ 0.5 \ 0.5 \ 0.5]$ is used. The procedure for optimizing the scaling factor is as follows: As a guideline provided by [1], alpha takes a small value in the first few decoding steps and increases as BER is better. This is to reduce the effect of the extrinsic information in the initial decoding steps when the BER is relatively high. Similarly, in our manual optimization of the alpha vector set, alpha is given to be a small value in the first decoding step. For the next decoding steps (iterations), alpha is increased by a small value and the BER generated by this alpha value is observed. Alpha is then increased gradually with the corresponding BER observed. This continues until there is an increase in BER (or drop in performance). The alpha chosen for this particular decoding step is the one which generates the best BER. The same procedure is then repeated for subsequent decoding steps. The result shows that with some optimization of the weighting factor, the simulated BER is able to approach the asymptotic BER bound as $E_b/N_0$ increases.
Fig. 5.5. Asymptotic BER bound and simulated BER values for PC(31, 26) in Pulsed Jamming AWGN Channel with $\rho = 0.3$ and $E_b/N_0 = 7\text{dB}$

Fig. 5.6 shows the asymptotic BER bound and simulated values (with full JSI and CSI) for PC(31, 26) in a pulsed jamming Rayleigh channel with $\rho = 0.5$, $E_b/N_0 = 10\text{dB}$ and 4 iterations. For both simulations (simulations which use the set of (1) manually optimised alpha values and (2) alpha values from (2.41) respectively), the normalised reliability factor from (5.37) is used. When the $\alpha$ weighting factor from (2.41) is used in the decoding, very poor BER performance is again obtained. Therefore, a set of manually optimized weighting factor given by $\alpha_{app} = [0.0, 0.2, 0.7, 0.7, 0.5, 0.5, 0.5, 0.5]$ is used, and the resultant BER approaches the asymptotic BER bound as $E_b/N_0$ increases.
Fig. 5.6. Asymptotic BER bound and its simulated values for PC(31, 26)² in Pulsed Jamming Rayleigh Fading Channel with $\rho = 0.5$, $E_b/N_0 = 7$dB

The above results show that with JSI and CSI applied to obtain the test positions and to compute the extrinsic information, decoding product codes using Pyndiah’s algorithm with optimized $\alpha$ values is able to approach the asymptotic BER bounds.
6.1 Conclusions

In this thesis, we have derived new weight enumerator expressions for conventional shortened and shortened-extended block/product codes, and shown that they are tight for Hamming, BCH and Golay product codes. The weight enumerator obtained is compact and easy to use as it is expressed in terms of known parameters of the original (unshortened) code. For small codes, the tightness of the proposed weight enumerator is verified using true weight enumerator values while for larger codes, the tightness is verified by deriving the associated BER bounds and comparing with Monte Carlo simulations.

To obtain better shortened codes than the conventional code shortening scheme (which nullifies a block of contiguous information symbol positions), we propose to shorten a code by nullifying carefully selected information symbol positions which correspond to codewords with the smallest Hamming distance, and we have devised an efficient algorithm to identify such information symbol positions using the parity check matrix. Called ordered shortening, the proposed scheme produces shortened codes with larger $d_{\min}$ (minimum Hamming distance) and/or smaller $A_d$ (code weight enumerator).
values than a conventional-shortened code with the same code length and code rate. These improved $d_{\text{min}}$ and $A_{\text{ms}}$ properties lead to improved BER performance for the ordered-shortened codes, as verified by computer simulations using the Pyndiah decoding algorithm. Furthermore, the $d_{\text{min}}$ values of the proposed ordered shortening scheme are shown to meet the theoretical upper bound values for some class of codes.

We have also extended the Pyndiah's turbo decoding algorithm for product code to pulsed jamming channels with AWGN and Rayleigh fading. The BER bound for pulsed jamming channel with Rayleigh fading is also derived. The results show that with the use of jamming state information and channel state information to obtain the test positions and to compute the extrinsic information, decoding small product codes using Pyndiah's algorithm is able to approach the asymptotic BER well. For large product codes, we found that the $\alpha$ weighting factor in Pyndiah's algorithm must be properly optimized in order to produce the best BER results.

### 6.2 Recommendations

Some suggestions for future work are given below:

a) As shown in Section 5.4, the values of the $\alpha$ weighting factors used for Pyndiah's algorithm for decoding larger product codes such as PC(31,26)$^2$ in pulsed jamming channels need to be re-optimized for a larger product code as the original values given in [1] cannot give satisfactory results. More investigations can be performed to derive the general expression of the optimized $\alpha$, or systematic ways to perform the optimization.
b) So far we have investigated Pyndiah's decoding of product codes (5.24) in pulsed jamming channels with high $E_b/N_0$ and $E_b/N_f$ values. More work should be done to extend Pyndiah's decoding algorithm to the low $E_b/N_f$ regions, which are also of interest, especially to military communications.

c) A shortening scheme (ordered shortening) for shortened code has been described in Chapter 4. Work can be extended to punctured code where a better puncturing scheme (as compared to the conventional puncturing scheme) can be designed.

d) The weight enumerator expressions derived in Chapter 3 are for conventional shortened and shortened-extended block/product codes. Work can be done to derive the weight enumerator for the ordered-shortened codes proposed in Chapter 4.
List of Publications


Bibliography


Appendix A - Verification of Derived Weight Spectrum for Shortened Hamming (31, 26)

Table A. Comparison of True and Approximated \( A_{d(k)} \) for Hamming (31, 26)

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Appendix B – Verification of Derived Weight Spectrum for Shortened BCH(63, 18)

Table B. Comparison of True and Approximated $A_{d,63}$ for BCH(63, 18)

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Appendix C – Derivation of (5.10)

Substituting (5.8) and (2.13) into (5.9),

\[ P_d(d, r_c | n_j) = \prod_{a_1} \prod_{a_d} \frac{1}{\pi} \int_{0}^{\sigma/2} e^{-\frac{\left( \frac{r_c E_b}{N_j + N_0} \sum_{a_i}^j + \frac{r_c E_b}{N_0} \sum_{i=n_j+1}^d a_i^2 \right)}{\sin^2 \phi}} d\phi \prod_{i=1}^{d} p(a_i) \, da_1 \cdots da_d \]

Since the fading coefficients \( a_i \) are i.i.d, the above expression can be reduced to a product of integral over each \( a_i \),

\[ P_d(d, r_c | n_j) = \frac{1}{\pi} \int_{0}^{\sigma/2} e^{-\frac{\left( \frac{r_c E_b}{N_j + N_0} \sum_{a_i}^j + \frac{r_c E_b}{N_0} \sum_{i=n_j+1}^d a_i^2 \right)}{\sin^2 \phi}} d\phi \prod_{i=1}^{d} p(a_i) \, da_1 \cdots da_d \]

\[ = \frac{1}{\pi} \int_{0}^{\sigma/2} \left[ \prod_{a_i} \frac{1}{\pi} \int_{0}^{\pi/2} e^{-\frac{\left( \frac{r_c E_b}{N_j + N_0} \sum_{a_i}^j + \frac{r_c E_b}{N_0} \sum_{i=n_j+1}^d a_i^2 \right)}{\sin^2 \phi}} d\phi \right] \prod_{i=1}^{d} p(a_i) \, da_1 \cdots da_d \]

Let

\[ U = \prod_{a_i} \frac{1}{\pi} \int_{0}^{\pi/2} e^{-\frac{\left( \frac{r_c E_b}{N_j + N_0} \sum_{a_i}^j + \frac{r_c E_b}{N_0} \sum_{i=n_j+1}^d a_i^2 \right)}{\sin^2 \phi}} d\phi \]

Each received symbol can be classified into one of 2 cases: jammed or unjammed.
Firstly, consider an unjammed symbol. $U$ becomes

$$U = \int_{a_i}^\phi \frac{r_E a_i^2}{N_b} e^{-\sin^2 \phi} \, da_i$$

with $p(a_i)$ from (2.14) which evaluates to $U_1$:

$$U_1 = \frac{\sin^2 \phi}{r_E + \sin^2 \phi}$$

Now consider a jammed symbol. $U$ becomes

$$U = \int_{a_i}^\phi \frac{r_E a_i^2}{N_b} e^{-\sin^2 \phi} \, da_i$$

which evaluates to $U_2$:

$$U_2 = \frac{\sin^2 \phi}{r_E + \sin^2 \phi}$$

Since there are $n_j$ jammed symbols and $(n - n_j)$ unjammed symbols,

$$P_2(d, r, n_j) = \frac{1}{\pi} \int_0^{\pi/2} \left[ \int_{a_i}^\phi \frac{r_E a_i^2}{N_b} e^{-\sin^2 \phi} \, da_i \right] \, d\phi$$

$$= \frac{1}{\pi} \int_0^{\pi/2} U_1^{d-n_j} U_2^{n_j} \, d\phi$$

$$= \frac{1}{\pi} \int_0^{\pi/2} \left( \frac{\sin^2 \phi}{r_E + \sin^2 \phi} \right)^{d-n_j} \left( \frac{\sin^2 \phi}{r_E + \sin^2 \phi} \right)^{n_j} \, d\phi$$

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Appendix D – Derivation of (5.21)

\[ \Lambda (d_j) = \ln \left( \frac{\sum_{j \in [\psi_j]} \exp \left[ \frac{-n}{2} \left( \frac{(\tau_j - c_i)^2}{\sigma_n^2 + \sigma_j^2 / \rho} \right) - \sum_{l=\eta_j+1}^{n} \frac{(\tau_l - c_i)^2}{2\sigma_n^2} \right]}{\sum_{j \in [\psi_j]} \exp \left[ \frac{-n}{2} \left( \frac{(\tau_j - c_i)^2}{\sigma_n^2 + \sigma_j^2 / \rho} \right) - \sum_{l=\eta_j+1}^{n} \frac{(\tau_l - c_i)^2}{2\sigma_n^2} \right]} \right) \]

= \ln \left( \frac{\sum_{j=1}^{n_g} \exp \left[ \frac{-n}{2} \left( \frac{(\tau_j - c_i)^2}{\sigma_n^2 + \sigma_j^2 / \rho} \right) - \sum_{l=\eta_j+1}^{n} \frac{(\tau_l - c_i)^2}{2\sigma_n^2} \right]}{\sum_{j=1}^{n_g} \exp \left[ \frac{-n}{2} \left( \frac{(\tau_j - c_i)^2}{\sigma_n^2 + \sigma_j^2 / \rho} \right) - \sum_{l=\eta_j+1}^{n} \frac{(\tau_l - c_i)^2}{2\sigma_n^2} \right]} \right) + \ln \left( \frac{\sum_{i=1}^{m} P_i}{\sum_{i=1}^{m} Q_i} \right)

= \sum_{j=1}^{n_g} \frac{(\tau_j - c_i)^2}{2\left( \frac{\sigma_n^2 + \sigma_j^2}{\rho} \right)} + \sum_{l=\eta_j+1}^{n} \frac{(\tau_l - c_i)^2}{2\sigma_n^2} + \ln \left( \frac{\sum_{i=1}^{m} P_i}{\sum_{i=1}^{m} Q_i} \right) \]

where
\[ P_i = \exp \left[ -\sum_{j=1}^{n_i} \frac{(\eta - c_i)^2}{2\sigma_n^2 + \sigma_j^2/\rho} - \sum_{l=n_j+1}^{n} \frac{(\eta - c_l')^2}{2\sigma_n^2} \right] \quad \text{with } c' \in \{s_{j^{-1}}\} \]

\[ Q_i = \exp \left[ -\sum_{j=1}^{n_i} \frac{(\eta - c_i)^2}{2\sigma_n^2 + \sigma_j^2/\rho} - \sum_{l=n_j+1}^{n} \frac{(\eta - c_l')^2}{2\sigma_n^2} \right] \quad \text{with } c' \in \{s_{j^{-1}}\} \]

\[ = \exp \left( \sum_{j=1}^{n_i} \frac{(\eta - c_i^{-10})^2 - (\eta - c_i)^2}{2(\sigma_n^2 + \sigma_j^2/\rho)} + \sum_{l=n_j+1}^{n} \frac{(\eta - c_l^{10})^2 - (\eta - c_l')^2}{2\sigma_n^2} \right) \]

and

\[ = \exp \left( \sum_{j=1}^{n_i} \frac{(\eta - c_i^{-10})^2 - (\eta - c_i)^2}{2(\sigma_n^2 + \sigma_j^2/\rho)} + \sum_{l=n_j+1}^{n} \frac{(\eta - c_l^{10})^2 - (\eta - c_l')^2}{2\sigma_n^2} \right) \]