DEGREE STRUCTURES BELOW $\mathbf{0}'$

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Contents

Summary

This thesis is concerned with various degree structures below $\mathbf{0}'$, varying from Turing degrees to truth-table degrees, from computably enumerable degrees to Δ^0_2 degrees. In Chapter 1, we first provide a general introduction to the development of computability theory in the last few decades, after which, we will present the motivation of our results contained in this thesis. Notation and terminology will be introduced briefly at the end of this chapter.

In Chapter 2, we consider the interaction between the cupping property and the high/low hierarchy. Cholak, Groszek and Slaman proved the existence of a c.e. degree cupping every low c.e. degree to a low degree, and claimed that for any nonzero c.e. degree $\mathbf{a}$, there is a nonhigh c.e. degree $\mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b}$, the supremum of $\mathbf{a}$ and $\mathbf{b}$, is high. Jockusch, Li and Yang proved that the c.e. degree $\mathbf{b}$ above can be low$_2$, and Wu proved that this degree can also be cappable. Our result in Chapter 2 shows that $\mathbf{b}$ above can even be a noncuppable degree, improving all the results along this line.

Chapters 3 and 4 are devoted to the infima of degrees at different levels in the Ershov hierarchy. Lachlan's nondiamond theorem shows that in $\mathbf{R}$, the structure of the computably enumerable degrees, if nonzero degrees $\mathbf{a}$ and $\mathbf{b}$ have supremum $\mathbf{0}'$, then they cannot have infimum $\mathbf{0}$. In contrast to this, Downey showed the existence of two nonzero d.c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with supremum $\mathbf{0}'$ and infimum $\mathbf{0}$ in the structure of d.c.e. degrees. That is, the diamond lattice can be embedded into the d.c.e. degrees preserving 0 and 1. This provides an elementary difference between $\mathbf{R}$ and $\mathbf{D}_2$. In his Ph.D. thesis, Wu provided another approach of such a lattice embedding by using the isolation pairs.

Lachlan observed that the infimum of two c.e. degrees in $\mathbf{R}$, if exists, coincides

with the infimum of these two degrees in the Δ_2^0 degrees. This coincidence is not true anymore in the d.c.e. degrees, as first noted by Kaddah.

In Chapter 3, we compare two approaches of diamond embeddings with the infimum considered in different structures. We will construct a pair of d.c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with supremum $\mathbf{0}'$ and infimum $\mathbf{0}$ in $\mathbf{D}_2$, but they do not have infimum $\mathbf{0}$ in this Δ_2^0 degrees. As the degrees constructed in Downey's construction also have infimum $\mathbf{0}$ in the Δ_2^0 degrees, our result shows that two approaches of diamond embeddings proposed by Downey and Wu are different in essence.

In Chapter 4, we consider the infima of d.c.e. degrees in $\mathbf{D}_2$ and $\mathbf{D}_3$, where $\mathbf{D}_3$ is the structure of 3-c.e. degrees. Kaddah proved that some d.c.e. degrees can have different infima in $\mathbf{D}_2$ and $\mathbf{D}_3$. Our work in Chapter 4 shows that such degrees occur densely in the c.e. degrees.

In Chapter 5, we consider the *almost universal cupping* property of d.c.e. degrees, where a d.c.e. degree $\mathbf{d}$ has the almost universal cupping property if $\mathbf{d}$ cups every c.e. degree not below it to $\mathbf{0}'$. The existence of incomplete d.c.e degrees with this property was shown by Cooper, Harrington, Lachlan, Lempp and Soare in 1991, where they proved the existence of an incomplete maximal d.c.e. degree. Their construction involves an extremely complicated $\mathbf{0}'''$-priority argument, and it is fairly hard to modify their construction to make the constructed degree have other properties. In this chapter, we introduce a $\mathbf{0}'''$-priority argument and show that there is an almost universal cupping degree $\mathbf{d}$ and a c.e. degree $\mathbf{a} < \mathbf{d}$ such that every incomplete c.e. degree comparable with $\mathbf{d}$ is below $\mathbf{a}$. Our result is new and our construction is much simpler than the one used by Cooper et al.

In Chapter 6, we explore the diamond embedding in the c.e. truth-table degree structure $\mathbf{R}_{tt}$. Jockusch and Mohrherr proved that the diamond lattice can be embedded into $\mathbf{R}_{tt}$ preserving 0 and 1 with two atoms low, where the construction is a rather clever finite injury argument. In this chapter, we will prove that such a lattice can be embedded into $\mathbf{R}_{tt}$ preserving 0 and 1, but with two atoms superhigh. Our construction is based on Jockusch and Mohrherr's diagonalization method, and will involve a subtle infinite injury argument.

Chapter 1

GENERAL INTRODUCTION

The effective or computable content of mathematics is a central topic in computability theory or recursion theory. The historical root and motivation of computable mathematics can be traced from Euclid for more than two millennia (see [84]). The modern computability theory is motivated by Hilbert's Program and originated from the famous Gödel's Incompleteness Theorem (see [36]), and was developed by Church, Kleene, Post, Turing, etc. As the foundation of computability theory, the formal definition of computability introduced by Turing in [86], called *Turing computability*, bases on a simple device — Turing machine, was enthusiastically accepted by the founders of this subject, Gödel, Church and Kleene, and now is the standard definition of computability. In fact, other significant versions of definition of computability, introduced by Gödel, Church, Kleene, Markov and Post respectively, had been proven to be equivalent by the end of 1940's.

By a classical result of Kleene, there are only countably many computable functions. Thus there are much more incomputable functions. Among them, the *Busy Beaver function*, which was introduced by the distinguished combinatoricist Tibor Radó, is a natural example of incomputable functions as it dominates all computable functions.

In computable mathematics, we are concerned with the problem of the existence of algorithm or effective computational procedure for solving various problems, which is called *decision problem* (see [21]). A positive solution to a decision problem consists of giving an algorithm for solving it, in this case, we say that the problem is *solvable*. A negative solution to a decision problem consists of showing that no algorithm for solving the problem exists, in this case, we say that the problem is *unsolvable*. Several decision problems are well-known to be unsolvable, such as the solvability of Diophantine equations, the decidability of true equations between words in a finitely presented group, the problem of determining the Kolmogorov complexity of a string and the word problem for certain formal languages, etc. These problems are Turing equivalent to the halting problem, whose Turing degree is denoted as $\mathbf{0}'$. It is natural to ask whether all the unsolvable problems are Turing equivalent to the halting problem. This is known as Post's Problem (see [70]), and it is this problem that initiated the study on the (sub)structures of Turing equivalence classes — Turing degrees.

Intuitively, a set A is Turing reducible to a set B if the membership of A can be effectively computed from the information of the membership of B, alternatively, we say A is computable in B, denoted by $A \leq_T B$. The Turing reduction gives rise to an equivalence relation on the power set of natural numbers, and the corresponding equivalence classes are called *Turing degrees*. Let $\mathbf{D}$ be the class of all Turing degrees. In computable mathematics, each known unsolvable problem corresponds to a computably enumerable set, where a set is called *computably enumerable* (c.e. for short) if its elements can be enumerated effectively. Especially, a set is called *computable* if both itself and its complement are computably enumerable. In fact, each solvable problem corresponds to a computable set. Accordingly, a degree is called a *computably enumerable degree* if it contains a computably enumerable set. Let $\mathbf{R}$ be the class of computably enumerable degrees, and $\leq$ be the partial order relation on the computably enumerable degrees induced by the Turing reduction $\leq_T$. It is easy to see that $\mathbf{R}$ is closed under join with respect to $\leq$, and has the least element $\mathbf{0}$, the Turing degree of any computable set, and the greatest element $\mathbf{0}'$, the Turing degree of the halting problem K. So the Post's Problem exactly asks whether all the incomputable c.e. sets have the same Turing degree as $\mathbf{0}'$. Friedberg [35], and Muchnik [64] independently, answered Post's problem affirmatively as follows.

Theorem 1.1. *(Friedberg [35], and Muchnik [64] independently) There are two c.e. sets A and B such that $A \not\leq_T B$ and $B \not\leq_T A$.*

In the proofs, they invented the so-called (finite) injury priority argument, a powerful technique in modern computability theory. This method and its developments by Shoenfield, Sacks, Lachlan, Lerman, Jockusch, Soare, and others, have been applied to analyze the effective content of results in a wide variety of mathematical fields including Algebra, Algebraic Topology, Analysis, Combinatorics, Logic, Mathematical Physicals, Model Theory, Orderings and Topology (see [32]). In this thesis, we will focus on developing such techniques and their applications on Logic. In the following of this chapter, we first recall some big breakthroughs on techniques and simultaneously go through the algebraic structure of Turing degrees.

By improving the method introduced in [35] and [64], Sacks [72] proved the following Splitting Theorem.

Theorem 1.2. *(Sacks Splitting Theorem [72]) For each incomputable c.e. degree* $\mathbf{a}$*, there are incomparable c.e. degrees below* $\mathbf{a}$ *with join* $\mathbf{a}$*.*

In 1961, Shoenfield [77] first proposed the infinite injury priority construction and proved a weak version of the thickness theorem. This method was also invented by Sacks independently in 1963 in his paper [71]. By the infinite injury priority argument, Sacks proved that the structure $(\mathbf{R}, \leq)$ is dense.

Theorem 1.3. *(Sacks Density Theorem [73]) Given two c.e. sets* $D <_T C$*, there exists a c.e. set* A *such that* $D <_T A <_T C$*.*

Harrington showed (see Shore and Slaman [79]) that Sacks Splitting Theorem and Sacks Density Theorem can be unified if the top degree of C is low$_2$.

Theorem 1.4. *(Harrington Splitting Theorem [79]) For any c.e. degrees* $\mathbf{a} < \mathbf{b}$*, if* $\mathbf{b}$ *is low*$_2$*, then there exist c.e. degrees* $\mathbf{x}, \mathbf{y}$ *such that* $\mathbf{a} \leq \mathbf{x} < \mathbf{b}$*,* $\mathbf{a} \leq \mathbf{y} < \mathbf{b}$ *and* $\mathbf{x} \cup \mathbf{y} = \mathbf{b}$*.*

However, in general, such a unification fails. This was proved by Lachlan in 1975.

Theorem 1.5. *(Lachlan Nonsplitting Theorem [48]) There exist c.e. degrees* $\mathbf{a} < \mathbf{b}$ *such that* $\mathbf{b}$ *is not splittable over* $\mathbf{a}$*. That is, for all c.e. degrees* $\mathbf{d_0}$ *and* $\mathbf{d_1}$*,*

$$[\mathbf{a} \leq \mathbf{d_0}, \mathbf{d_1} \leq \mathbf{b} \ \&\ \mathbf{b} \leq \mathbf{d_0} \cup \mathbf{d_1}] \Rightarrow [\mathbf{b} \leq \mathbf{d_0} \vee \mathbf{b} \leq \mathbf{d_1}].$$

Lachlan nonsplitting theorem is a fairly significant step in the history of computability theory as it introduced a powerful technique, the $\mathbf{0}'''$-priority construction, a technique was referred as the "monster method" at that time because of its apparent great complexity. This theorem was extended by Harrington in 1980.

Theorem 1.6. *(Harrington Nonsplitting Theorem [38]) There exists an incomplete c.e. degree* $\mathbf{a}$ *above which* $\mathbf{0}'$ *is not splittable. That is, there are no c.e. degrees* $\mathbf{x}, \mathbf{y}$ *such that* $\mathbf{a} < \mathbf{x} < \mathbf{0}', \mathbf{a} < \mathbf{y} < \mathbf{0}'$ *and* $\mathbf{x} \cup \mathbf{y} = \mathbf{0}'$.

After seeing Sacks Splitting and Density Theorems, Shoenfield conjectured in [78] that $(\mathbf{R}, \leq)$ is a dense structure as an upper semi-lattice analogous to the rationals being a dense structure as a linearly ordered set. From this conjecture, he deduced the following two assertions:

(i) If $\mathbf{a}, \mathbf{b} \in \mathbf{R}$ are incomparable, then they have no infimum in $\mathbf{R}$.

(ii) Given c.e. degrees $\mathbf{0} < \mathbf{b} < \mathbf{a}$, there exists a c.e. degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{b} \cup \mathbf{c}$.

Assertion (i), and hence, Shoenfield's conjecture, was first refuted by Lachlan [46] and Yates [91] independently, who proved the existence of two incomparable c.e. degrees with infimum $\mathbf{0}$.

Theorem 1.7. *(Lachlan [46] and Yates [91] independently) There are two incomparable c.e. sets* A *and* B *such that if* $C \leq_T A$ *and* $C \leq_T B$ *then* C *is computable. Such a pair of* A *and* B *is called a minimal pair.*

A c.e. degree $\mathbf{a}$ is *cappable*, if $\mathbf{a}$ is either $\mathbf{0}$ or $\mathbf{a}$ is one part of a minimal pair. $\mathbf{a}$ is called *noncappable* if it is not cappable. Let $\mathbf{M}$ denote the class of cappable degrees and $\mathbf{NC}$ denote the class of noncappable degrees. $\mathbf{NC}$ and $\mathbf{M}$ form a proper partition of $\mathbf{R}$, where the existence of noncappable degrees was first proved by Yates in [91].

Dually, a c.e. degree $\mathbf{a}$ is *cuppable*, if there is an incomplete c.e. degree $\mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b}$, the supremum of $\mathbf{a}$ and $\mathbf{b}$, is $\mathbf{0}'$. Moreover, a c.e. degree $\mathbf{a}$ is *low-cuppable*, if there is a low c.e. degree $\mathbf{b}$ cupping $\mathbf{a}$ to $\mathbf{0}'$. A c.e. degree $\mathbf{a}$ is *noncuppable* if it is not cuppable. Assertion (ii) implies that every incomputable degree is cuppable, which is also not true, by Yates' work on the existence of noncuppable degrees. We use **LC** and **NCup** to denote the class of low-cuppable degrees and the class of noncuppable degrees, respectively.

Theorem 1.8. *(Yates unpublished (cf. Cooper [15])) There is an incomplete c.e. degree* $\mathbf{a}$ *such that for any c.e. degree* $\mathbf{b}$*, if* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$*, then* $\mathbf{b} = \mathbf{0}'$*.*

Just as expected by Shoenfield himself when he proposed this conjecture, even though his conjecture is not true, it plays a significant role in the development of computability theory, not only because of its impact on the structural properties of c.e. degrees, but also because of the techniques invented to solve these problems.

A coinfinite c.e. set A is called *promptly simple*, PS for short, if there is a partial computable function p and a computable enumeration $\{A_s\}_{s \in \omega}$ of A such that for every e, if W_e is infinite, then there are x, s such that $x \in W_{e,\text{at } s} \cap A_{p(s)}$. A c.e. degree is called promptly simple or PS if it contains a promptly simple set. Let **PS** denote the class of all promptly simple degrees.

In 1984, Ambos-spies, Jockusch, Shore and Soare proved in [1] that $\mathbf{M}$, the class of cappable degrees, is also an ideal of $\mathbf{R}$, and hence the ideal of noncuppable degrees is a subideal of $\mathbf{M}$.

Theorem 1.9. *(Ambos-spies, Jockusch, Shore and Soare [1]) In the structure of c.e. degrees* $\mathbf{R}$*, we have*

(1) $\mathbf{M}$ *is an ideal in* $\mathbf{R}$*.*

(2) $\mathbf{NC} = \mathbf{PS} = \mathbf{LC}$ *is a strong filter in* $\mathbf{R}$*.*

This paper initiated the research on the relation between the cupping property and the high/low hierarchy. Obviously, **NCup** is an ideal of $\mathbf{R}$. Harrington proved that every c.e. degree is either cappable or cuppable, and that there is a cuppable degree in $\mathbf{M}$. Thus, **NCup** is a proper subset of $\mathbf{M}$.

In 1978, Harrington proposed a much stronger cupping property, called plus-cupping property, where a nonzero c.e. degree $\mathbf{a}$ is a *plus-cupping degree* (following Fejer and Soare [34]) if every nonzero c.e. degree below $\mathbf{a}$ is cuppable to $\mathbf{0}'$.

Theorem 1.10. *(Harrington Plus-cupping Theorem) Plus-cupping degrees exist.*

Harrington proved in [37] that every high c.e. degree bounds a high noncuppable degree. The construction is fully presented by Miller in [61]. One significant application of the cupping/noncupping properties were done by Harrington and Shelah in [39], who proved that the first order theory of the structure $(\mathbf{R}, \leq)$ is undecidable.

In 1993, Ambos-Spies, Lachlan and Soare showed that there is no minimal cuppable degrees.

Theorem 1.11. *(Ambos-Spies, Lachlan and Soare [2]) For any given incomplete c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$*, if* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$*, then there is a c.e. degree* $\mathbf{c}$ *below* $\mathbf{a}$ *cupping* $\mathbf{b}$ *to* $\mathbf{0}'$*.*

After Lachlan observed that every incomputable c.e. set has a major subset, he proposed in [47] the concept of major subdegrees analogous to the notion of major subsets as follows: for two c.e. degrees $\mathbf{b} < \mathbf{a}$, $\mathbf{b}$ is called *a major subdegree* of $\mathbf{a}$ if each c.e. degree cupping $\mathbf{a}$ to $\mathbf{0}'$ also cups $\mathbf{b}$ to $\mathbf{0}'$. Considering the cuppability of c.e. degrees, Lachlan asked whether every c.e. degree $\mathbf{a}$ between $\mathbf{0}$ and $\mathbf{0}'$ has a major subdegree? Restricted to the low$_2$ c.e. degrees, Seetapun gave an affirmative answer to this question.

Theorem 1.12. *(Seetapun [75]) Each incomputable low$_2$ c.e. degree has a major subdegree.*

Cooper and Li announced recently in [19] a full solution to Lachlan's major subdegree problem.

A c.e. degree $\mathbf{a}$ is called *low$_n$-cuppable* by Li, Wu and Zhang in [53], if there is a low$_n$ c.e. degree $\mathbf{b}$ cupping $\mathbf{a}$ to $\mathbf{0}'$. Obviously, every low-cuppable degree is low$_n$-cuppable. The converse is not true.

Theorem 1.13. *(Li, Wu and Zhang [53]) There is a low$_2$-cuppable degree that is not low-cuppable.*

Recent work of Greenberg, Ng and Wu shows the existence of a cuppable degree, which can only be cupped to $\mathbf{0}'$ by a high degree. Therefore, for any $n \in \omega$, the set of low$_n$-cuppable degrees is a proper subset of the cuppable degrees.

In [7], Bickford and Mills introduced the notion of deep degree, where a c.e. degree $\mathbf{a}$ is *deep* if $(\mathbf{a} \cup \mathbf{b})' = \mathbf{b}'$ for any c.e. degree $\mathbf{b}$. Note, the deep degrees form an ideal of $\mathbf{R}$. However, Lempp and Slaman proved in [50] that this ideal is trivial.

Theorem 1.14. *(Lempp and Slaman [50]) No incomputable deep degree exists.*

By salvaging the idea of the deep degree, in [11], Cholak, Groszek and Slaman introduced the concept of the almost deep degree as follows: a c.e. degree is called *almost deep* if it cups every low c.e. degree to a low degree. It is easy to see that the almost deep degrees form a proper subideal of $\mathbf{M}$. Furthermore,

Theorem 1.15. *(Cholak, Groszek and Slaman [11]) There is a nonzero almost deep degree.*

In the same paper, Cholak, et al. pointed out that every nonzero c.e. degree can cup a low$_2$ c.e. degree to a non-low$_2$ degree, and that for each nonzero c.e. degree $\mathbf{a}$, there is a nonhigh c.e. degree $\mathbf{b}$ such that $\mathbf{b}$ cups $\mathbf{a}$ to a high degree. In [42], Jockusch, Li and Yang proved that these two results can be unified.

Theorem 1.16. *(Jockusch, Li and Yang [42]) Every nonzero c.e. degree* $\mathbf{c}$ *is cuppable to a high c.e. degree by a low*$_2$ *c.e. degree* $\mathbf{b}$. *Hence, in terms of the high/low hierarchy, Cholak, Groszek and Slaman's result is the best possible.*

It is natural to ask in which subclass of low$_2$ c.e. degrees can $\mathbf{b}$ in [42] be located. Wu proved in [89] that such a $\mathbf{b}$ can be cappable. In Chapter 2, we prove that $\mathbf{b}$ in Theorem 1.16 can also be selected as a noncuppable degree.

Theorem 1. *(Liu and Wu [56]) Given an incomputable c.e. degree* $\mathbf{c}$*, there is a noncuppable low*$_2$ *c.e. degree* $\mathbf{b}$ *such that* $\mathbf{c} \cup \mathbf{b}$ *is high.*

We comment here that in Theorem 1, if $\mathbf{c}$ is incomplete, then $\mathbf{b} \cup \mathbf{c}$ is also incomplete. This incompleteness cannot be obtained in Jockusch, Li and Yang's construction.

$\mathbf{R}$, the class of computably enumerable degrees, is located at the first level in the Ershov hierarchy, while $\mathbf{D}_2$, the class of d.c.e. degrees, those Turing degrees containing differences of c.e. sets, is located at the second level. Most constructions of c.e. degrees can be used to construct d.c.e. degrees. For instance, in [12], Cooper was able to construct a proper d.c.e. degree, a d.c.e. degree containing no c.e. sets by using a finite injury construction. Therefore, $\mathbf{R}$ is a proper subset of $\mathbf{D}_2$.

The early work showed that $\mathbf{D}_2$ shares many structural properties with $\mathbf{R}$. A notable observation of Lachlan says that any nonzero d.c.e. degree bounds a nonzero c.e. degree, and hence $\mathbf{D}_2$ is downwards dense. The first two structural differences between $\mathbf{D}_2$ and $\mathbf{R}$ were obtained by Arslanov and Downey.

Theorem 1.17. *(Arslanov's Cupping Theorem [3]) Every nonzero d.c.e. degree cups to $\mathbf{0}'$ with an incomplete d.c.e. degree. That is, every d.r.e degree is cuppable in $\mathbf{D}_2$.*

Theorem 1.18. *(Downey Diamond Theorem [25]) The diamond lattice can be embedded into the d.c.e. degrees preserving $\mathbf{0}$ and $\mathbf{1}$.*

Cooper, et al. proved in [16] that the d.c.e. degrees are not dense by showing there is a maximal element. This gives another elementary difference between $\mathbf{D}_2$ and $\mathbf{R}$.

Theorem 1.19. *(Cooper, Harrington, Lachlan, Lempp and Soare [16]) There is an incomplete d.c.e. degree $\mathbf{d}$ such that the interval $(\mathbf{d}, \mathbf{0}')$ contains no d.c.e. degree. Consequently, $\mathbf{D}_2$ is not densely ordered.*

Cooper et al.'s construction of incomplete maximal d.c.e. degrees is fairly complicated, and leaves very little freedom to combine it with other strategies. For instance, we don't know whether there are two such incomplete maximal degrees forming a minimal pair. If it is true, then it will imply both Theorem 1.17 and Theorem 1.18. The following theorem says that no incomplete maximal d.c.e. degrees can be low.

Theorem 1.20. *(Arslanov, Cooper and Li [4, 5]) For any given low d.c.e. degree $\mathbf{l}$, there is a low d.c.e. degree $\mathbf{c}$ above $\mathbf{l}$.*

Even though the structure of $\mathbf{D}_2$ is not dense, various weak density properties have been investigated after Cooper et al.'s work.

Theorem 1.21. *(Cooper and Yi [20], Ishmukhametov [40] independently) For any c.e. degree* $\mathbf{a}$ *and d.c.e. degree* $\mathbf{d} > \mathbf{a}$*, there is a d.c.e. degree* $\mathbf{c}$ *such that* $\mathbf{a} < \mathbf{c} < \mathbf{d}$*.*

The following is a direct extension of Theorem 1.21.

Theorem 1.22. *(Arslanov, LaForte and Slaman [6]) For any c.e. set* B *and* n*-c.e. set* A *with* $n \geq 1$ *and* $B <_T A$*, there is a d.c.e. set* C *with* $B <_T C <_T A$*.*

In Theorem 1.21, the requirement " $\mathbf{c}$ is d.c.e." is necessary.

Theorem 1.23. *(Cooper and Yi [20]) There exist a c.e. degree* $\mathbf{a}$ *and a d.c.e. degree* $\mathbf{d}$ *such that* $\mathbf{a} < \mathbf{d}$ *and* $\mathbf{a}$ *is the greatest c.e. degree below* $\mathbf{d}$*.*

According to the notion introduced in [20], $\mathbf{a}$ is said to *isolate* $\mathbf{d}$ and $(\mathbf{a}, \mathbf{d})$ is called an isolation pair. The existence of isolation pairs, as pointed out by Cooper and Yi in [20] in a footnote, can also be obtained from a result of Kaddah.

Theorem 1.24. *(Kaddah [44]) Every low c.e. degree is branching in the d.c.e. degrees, where a degree* $\mathbf{a}$ *is branching in the d.c.e. degrees means that there are two d.c.e. degrees* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *strictly above* $\mathbf{a}$ *such that for each d.c.e. degree* $\mathbf{c} \leq \mathbf{b}_1, \mathbf{b}_2$*,* $\mathbf{c} \leq \mathbf{a}$*.*

It seems that if $(\mathbf{a}, \mathbf{d})$ is an isolation pair, then they should be close to each other. However, this is not true in the sense of the high/low hierarchy.

Theorem 1.25. *(Ishmukhametov and Wu [41]) There is an isolation pair* $(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{a}$ *is low c.e. and* $\mathbf{d}$ *is high d.c.e..*

In [87], Wu provided another proof of Theorem 1.18.

Theorem 1.26. *(Wu [87]) There are an isolation pair* $(\mathbf{a}, \mathbf{d})$ *and a c.e. degree* $\mathbf{c}$ *such that* $\mathbf{c}$ *cups* $\mathbf{d}$ *to* $\mathbf{0}'$*, and caps* $\mathbf{a}$ *(in the d.c.e. degrees) to* $\mathbf{0}$*. Obviously,* $\mathbf{c}$ *also caps* $\mathbf{d}$ *to* $\mathbf{0}$ *in the d.c.e. degrees, and hence,* $\{\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ *is a diamond embedding into the d.c.e. degrees.*

The following proposition is used to prove Theorem 1.26.

Proposition 1.27. *Let* $(\mathbf{a}, \mathbf{d})$, $(\mathbf{c}, \mathbf{e})$ *be two isolation pairs with* $\mathbf{c} \cap \mathbf{a} = \mathbf{0}$, *then* $\mathbf{d}$ *and* $\mathbf{e}$ *form a minimal pair in the d.c.e. degrees.*

Lachlan proved in [46] that two c.e. degrees form a minimal pair in the c.e. degrees if and only if they form a minimal pair in the Δ^0_2 degrees. Kaddah proved that this property cannot be generalized to the d.c.e. degrees.

Theorem 1.28. *(Kaddah [44]) There are two d.c.e. degrees* $\mathbf{a}, \mathbf{b}$ *and an* ω*-c.e. degree* $\mathbf{e}$ *such that* $\mathbf{0} < \mathbf{e} < \mathbf{a}, \mathbf{b}$ *and* $\mathbf{a}, \mathbf{b}$ *form a minimal pair in the d.c.e. degrees.*

In [55], Liu and Wu gave another proof of Theorem 1.28.

Theorem 2. *(Liu and Wu [55]) There are c.e. degrees* $\mathbf{a}, \mathbf{b}$*, d.c.e. degrees* $\mathbf{c}, \mathbf{d}$ *and a nonzero* ω*-c.e. degree* $\mathbf{e} \leq \mathbf{c}, \mathbf{d}$ *such that* $\mathbf{a}, \mathbf{b}$ *form a minimal pair, and that* $(\mathbf{a}, \mathbf{c})$, $(\mathbf{b}, \mathbf{d})$ *are isolation pairs. Thus, as mentioned in Proposition 1.27,* $\mathbf{c}, \mathbf{d}$ *form a minimal pair in the d.c.e. degrees, but not in the* Δ^0_2 *degrees.*

The construction in Theorem 2 can be incorporated with the splitting of $\mathbf{0}'$ and hence we have the following diamond embedding theorem.

Theorem 3. *(Liu and Wu [55]) There are c.e. degrees* $\mathbf{a}, \mathbf{b}$*, d.c.e. degrees* $\mathbf{c}, \mathbf{d}$ *and a nonzero* ω*-c.e. degree* $\mathbf{e} \leq \mathbf{c}, \mathbf{d}$ *such that* $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$, $\mathbf{a}, \mathbf{b}$ *form a minimal pair, and that* $(\mathbf{a}, \mathbf{c})$, $(\mathbf{b}, \mathbf{d})$ *are isolation pairs. Consequently,* $\{\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ *form a diamond in the d.c.e. degrees, but not in the* Δ^0_2 *degrees.*

We note that the two atoms in Downey's diamond embedding form a minimal pair in the Δ^0_2 degrees, not just in the d.c.e. degrees. Therefore, Theorem 3 shows that the two approaches proposed by Downey and by Wu are different in essence.

In [44], Kaddah proved that the infima of some d.c.e. degrees can even be different in d.c.e. degrees and 3-c.e. degrees.

Theorem 1.29. *(Kaddah [44]) There are d.c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and a 3-c.e. degree* $\mathbf{x}$ *such that* $\mathbf{a}$ *is the infimum of* $\mathbf{b}, \mathbf{c}$ *in the d.c.e. degrees, but not in the 3-c.e. degrees, as* $\mathbf{a} < \mathbf{x} < \mathbf{b}, \mathbf{c}$.

In Chapter 4, we prove that such 4-tuples $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x})$ occur densely in the c.e. degrees.

Theorem 4. *(Liu, Wang and Wu [54]) Given c.e. degrees* $\mathbf{u} < \mathbf{v}$*, there are d.c.e. degrees* $\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}$ *and a 3-c.e. degree* $\mathbf{x}$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{a} < \mathbf{x} < \mathbf{b_1}, \mathbf{b_2}$*, and at the same time,* $\mathbf{b_1}$ *and* $\mathbf{b_2}$ *have infimum* $\mathbf{a}$ *in the d.c.e. degrees.*

We also consider cupping properties in the Turing degrees below $\mathbf{0}'$. Posner and Robinson proved in [69] that any incomputable degree $\mathbf{a}$ below $\mathbf{0}'$ is complemented in the Δ^0_2 degrees, where a degree $\mathbf{a}$ is complemented if there is a degree $\mathbf{b}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$. Slaman and Steel were able to show that any incomputable degree $\mathbf{a}$ below $\mathbf{0}'$ has a 1-generic degree as its complement.

Theorem 1.30. *(Slaman and Steel [82]) Each nonzero degree* $\mathbf{a}$ *below* $\mathbf{0}'$ *has a 1-generic complement.*

In [76], Seetapun and Slaman showed that the complements can be minimal degrees.

Theorem 1.31. *(Seetapun and Slaman [76]) Any nonzero c.e. degree has a minimal degree as its complement.*

Cooper and Seetapun, and independently Li, announced the existence of universal cupping degrees, where an incomplete Δ^0_2 degree has the universal cupping property if it cups each nonzero c.e. degree to $\mathbf{0}'$.

Theorem 1.32. *(Lewis [51]) Universal cupping degrees exist and can be minimal degrees.*

Furthermore, Li, Song and Wu [52] proved that universal cupping degrees can be ω-c.e. By Lachlan's observation that every incomputable d.c.e. degree bounds an incomputable c.e. degree, no universal cupping degree can be n-c.e. We are interested in those incomplete d.c.e. degrees cupping all c.e. degrees not below itself to $\mathbf{0}'$. Such degrees are called *almost universal cupping degrees.* Obviously, each incomplete maximal d.c.e. degrees is an almost universal cupping degree. However, as mentioned before, due to the hard construction introduced in Cooper et al.'s paper, very little extra properties are known about such incomplete maximal d.c.e. degrees. In Chapter 5, we prove that almost universal cupping degrees can have greatest c.e. predecessors.

Theorem 5. *(Liu and Wu [57]) There are an incomplete d.c.e. set D and a c.e. set B such that $D \not\leq_T B$, and for any c.e. set W_e, either $B \oplus D \oplus W_e \equiv_T \emptyset'$ or $W_e \leq_T B$.*

Let $\mathbf{b} = deg_T(B)$ and $\mathbf{d} = deg_T(B \oplus D)$. Theorem 5 says that $\mathbf{d}$ is not only an almost universal cupping degree, but also every c.e. degree which $\mathbf{d}$ does not cup to $\mathbf{0}'$ is bounded by $\mathbf{b}$. Note that by Theorem 1.21 and Theorem 5, the class of maximal d.c.e. degrees is a proper subclass of the almost universal cupping d.c.e. degrees.

In Chapter 6, we will explore the structure of truth-table degrees, a degree structure induced by truth-table reduction. Here, A is *truth-table reducible* to B, denoted by $A \leq_{tt} B$, if there is a total computable function f such that $x \in A \iff B \models \sigma_{f(x)}$. The following is an equivalent definition of truth-table reduction in terms of functionals, which shows a difference between truth-table reductions and Turing reductions.

Theorem 1.33. *(Trakhtenbrot [85] and Nerode [65] independently) $A \leq_{tt} B$ iff $A = \Phi^B$ for some total Turing functional Φ, where a Turing functional Φ is called total if Φ^X is a total function for each $X \subseteq \omega$.*

We are interested in those sets truth-table reducible to K. It is well-known that a set X is truth-table reducible to K if and only if it is ω-c.e. Say that a tt-degree $\mathbf{a}$ is c.e. if it contains a c.e. set. We let $\mathbf{R}_{tt}$ denote the class of all c.e. tt-degrees, and $\mathbf{D}_{tt}$ denote the class of all tt-degrees. While $\mathbf{D}_{tt}(\geq \mathbf{0}')$, the tt-degrees greater than $\mathbf{0}'$, is dense (see Mohrherr [62]), $\mathbf{R}_{tt}$ is not dense, as in $\mathbf{R}_{tt}$, nonzero minimal elements ([22], [58] and [33]) exist. Marchenkov [59] proved that there is a nontrivial upper bound for all the minimal c.e. tt-degrees. With regard to the infima of c.e. tt-degrees, each incomplete c.e. tt-degree is branching in $\mathbf{R}_{tt}$ and each Turing incomplete c.e. tt-degree is a part of a minimal pair in $\mathbf{R}_{tt}$. Jockusch and Mohrherr proved in [43] that the diamond lattice can be embedded into $\mathbf{R}_{tt}$ preserving 0 and 1. By Lachlan Nondiamond Theorem, we know that this cannot be true in $\mathbf{R}$.

Theorem 1.34. *(Jockusch and Mohrherr [43]) The diamond lattice can be embedded into $\mathbf{R}_{tt}$ preserving 0 and 1.*

As Jockusch and Mohrherr pointed out in [43], the construction in [43] can easily be modified to make the two atoms of the diamond low. It is natural to ask whether those two atoms can be high, or even superhigh, where a c.e. set A is *superhigh* if $\emptyset'' \leq_{tt} A'$. The notion of superhigh degrees was first introduced by Mohrherr in [63]. She proved the existence of incomplete superhigh c.e. degrees, and also high, but not superhigh, c.e. degrees. More recently, Ng [66] proved that there is a minimal pair (in the Turing degrees) of superhigh c.e. degrees, a result announced by Shore earlier.

Recent research shows that the notion of superhighness is closely related to effective measure theory. To investigate the relationship between degree theory and measure theory, Dobrinen and Simpson [24] introduced the notion of almost everywhere domination, where a set A is *almost everywhere dominating* (a.e.d. for short) if for almost all $X \subseteq \omega$ and all $g \leq_T X$, there exists a function $f \leq_T A$ such that f dominates g. Martin's result in [60] implies that the a.e.d. degrees must be high (see [24]), and Cholak, Greenberg and Miller constructed an incomplete a.e.d. c.e. degree. In [8], Binns, Kjos-Hanssen, Lerman and Solomon showed that not every high degree is a.e.d. This result actually follows Simpson's result ([81]) that every a.e.d. degree must be superhigh and that not every superhigh degree is a.e.d.

In Chapter 6, we will prove the diamond lattice can be embedded into $\mathbf{R}_{tt}$ preserving 0 and 1 with superhigh atoms.

Theorem 6. *(Cenzer, Franklin, Liu and Wu [10]) There are superhigh computably enumerable sets A and B such that $\mathbf{0}_{tt}$, $deg_{tt}(A)$, $deg_{tt}(B)$, and $\mathbf{0}'_{tt}$ form a diamond in the c.e. tt-degrees.*

Our proof involves a subtle infinitary argument due to the interaction between the superhigh and minimal pair strategies.

In the following of this chapter, we present the notation and terminology which will be needed in this thesis.

Basic notation and terminology

Our notation and terminology are standard and generally follow Odifreddi [67, 68] and Soare [83].

We will deal with sets and functions over the natural numbers $\omega = \{0, 1, 2, \dots\}$. As usual, we reserve lower-case letters $a, b, c, d, e, i, j, k, \dots, x, y, z$ for integers and upper-case letters $A, B, C, \dots, X, Y, Z$ for subsets of ω. f, g, h, and occasionally, some other lower-case letters, denote total functions from ω^n to ω, for $n \geq 1$. We will use lower-case Greek letters, such as φ, ψ, θ, etc. to denote partial functions and upper-case Greek letters, such as Φ, Ψ, Θ etc. to denote functionals.

Given a function φ, $\varphi(x)\downarrow$ denotes that $\varphi(x)$ is defined and $\varphi(x)\downarrow= y$ denotes that $\varphi(x)$ is defined and has value y. $\varphi(x)\uparrow$ denotes that $\varphi(x)$ is undefined. For partial functions φ and ψ, $\varphi = \psi$ means that for all $x \in \omega$, $\varphi(x)\downarrow$ iff $\psi(x)\downarrow$, and if $\varphi(x)\downarrow$ then $\varphi(x) = \psi(x)$. For a function f, $f \upharpoonright x$ denotes the restriction of f to arguments $y < x$. $\langle x, y\rangle$ denotes the image of (x, y) under the standard paring function $\frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$ from $\omega \times \omega$ onto ω.

We use 2^ω, and sometimes, $\mathcal{P}(\omega)$, to denote the power set of ω and $2^{<\omega}$ to denote the set of finite 0-1-strings. Similarly, we use ω^ω to denote the set of functions from ω to ω and $\omega^{<\omega}$ to denote the set of finite strings of natural numbers. We will use $\alpha, \beta, \dots$ to denote finite strings. For a string α, $lh(\alpha)$ denotes the least number not in the domain of α, and we will call $lh(\alpha)$ the length of string α. For strings α and β, $\alpha^\frown\beta$ denotes the concatenation of α and β, and $\alpha \subseteq \beta$ denotes that α is an initial segment of β. Similarly, $\alpha \subseteq f$ denotes that α is an initial segment of f.

Intuitively, a function φ is *computable or recursive,* if there is a program P such that for all $n \in \omega$, if $P(n)$ is defined, then $\varphi(n)$ converges with outcome equal to $P(n)$, and if $P(n)$ is not defined, then $\varphi(n)$ diverges. As there are countably many programs, there are only countably many partial computable functions and these functions can be effectively listed in a uniform way as follows

$$\varphi_0,\ \varphi_1,\ \dots,\ \varphi_e,\ \dots$$

where $e \in \omega$. We denote $dom(\varphi_e)$ by W_e. A set A is *computably enumerable*, c.e. for short, if A is the domain of a partial computable function. We have an effective list of c.e. sets and we call W_e the e-th c.e. set.

Let $K = \{e : e \in W_e\}$, and K is called the *halting problem*. Obviously, K is a c.e. set and all c.e. sets are Turing reducible (1-reducible, to be precise) to K, which implies that the Turing degree of K is the greatest element in the c.e. degrees.

We write $\varphi_{e,s}(x) = y$ if $e, x, y < s$ and on input x, the program φ_e stops in s steps with y as the output. In this case, we say that $\varphi_e(x)$ converges in s steps, and we write $\varphi_{e,s}(x)\downarrow$ to denote this convergence. We write $\varphi_e(x)\downarrow$ to denote that $\varphi_e(x)$ converges, i.e. $\varphi_{e,s}(x)\downarrow$ for some s. Accordingly, we write $\varphi_e(x)\downarrow= y$ if $\varphi_e(x)$ converges with output y. Correspondingly, let $W_{e,s}$ be the set $\{x : \varphi_{e,s}(x)\downarrow\}$, and we write $W_{e,\text{at } s}$ for $W_{e,s} - W_{e,s-1}$ if $s > 0$.

Given sets A and B, we say that A *is computable in* B, or A *is Turing reducible to* B, denoted by $A \leq_T B$, if there is an algorithm, or a program, with B as an oracle, by which the membership of A can be effectively computed from the membership of B. Let Φ be the algorithm in the Turing reduction $A \leq_T B$, where Φ is called *Turing functional*. Now, we write $A = \Phi^B$ to denote the reduction $A \leq_T B$ via Turing functional Φ explicitly. Then we can effectively list all the Turing functionals as

$$\Phi_0,\ \Phi_1,\ \ldots,\ \Phi_e, \ldots.$$

So if $A \leq_T B$, then there is some e such that $A = \Phi_e^B$. Turing reduction $\leq_T$ is a partial order on 2^ω, and we can define an equivalence relation $\equiv_T$ on 2^ω by $A \equiv_T B \iff A \leq_T B \ \wedge\ B \leq_T A$, in a natural way. This equivalence relation is called *Turing equivalence*, and the equivalence classes under $\equiv_T$ are called *Turing degrees*, which are denoted by boldface lower-case letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$. Given a set A, its Turing degree $\mathbf{a}$ is also denoted by $deg_T(A)$. Let $\mathbf{D}$ be the class of all Turing degrees, and for all $\mathbf{a}, \mathbf{b} \in \mathbf{D}$, we define $\mathbf{a} \leq \mathbf{b}$ if $A \leq_T B$ for some $A \in \mathbf{a}$ and $B \in \mathbf{b}$. A degree $\mathbf{a}$ is c.e. if it contains a c.e. set. We let $\mathbf{R}$ denote the class of c.e. degrees.

The *jump operator*, $'$, is an operator from 2^ω to 2^ω defined by $X' = \{e : \Phi_e^X(e)\downarrow\}$ for $X \subseteq \omega$. X' is the *Turing jump* of X. In particular, $\emptyset' = K$ and for any X, $X <_T X'$. By induction on n, we can define the *n-th jump* of X, denoted by $X^{(n)}$, as the Turing jump of $X^{(n-1)}$. Accordingly, for a given degree $\mathbf{a}$, the Turing jump of $\mathbf{a}$, denoted by $\mathbf{a}'$, is defined as the Turing degree of A', where A is any set in $\mathbf{a}$. This induced jump operator is well-defined on $\mathbf{D}$, and provides a natural classification of Turing degrees below $\mathbf{0}'$. A degree $\mathbf{a} \leq \mathbf{0}'$ is *high*$_n$ if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$, and a degree $\mathbf{a} \leq \mathbf{0}'$ is *low*$_n$ if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$. *High*$_1$ degrees and *low*$_1$ degrees are also called *high* degrees and *low* degrees, respectively. This classification is now known as the *high/low hierarchy*. Maritin [60], Lachlan [45], and Sacks [74] each showed that there are c.e. degrees neither high$_n$ nor low$_n$ for all $n \in \omega$.

Shoenfield Limit Lemma says that a set D is Turing reducible to K if and only if there is a computable approximation of D, $\{D_s\}_{s\in\omega}$, such that $D_0 = \emptyset$, each D_s is finite and for any number x, $D(x) = \lim_s D_s(x)$. In terms of this, a c.e. set A can be regarded as a set with an effective approximation $\{A_s\}_{s\in\omega}$ such that $A_0 = \emptyset$ and for any x, $A(x) = \lim_s A_s(x)$ with $|\{s : A_s(x) \neq A_{s+1}(x)\}| \leq 1$. This idea can be generalized to approximate a special subclass of Δ^0_2 sets. We say that a set B is ω-c.e. if there is a computable function f such that $|\{s : B_s(x) \neq B_{s+1}(x)\}| \leq f(x)$ for all $x \in \omega$. In particular, B is called an n-c.e. set if $f(x) = n$ for all $x \in \omega$. Accordingly, we say that a Turing degree $\mathbf{x}$ is n-c.e. (ω-c.e.) if $\mathbf{x}$ contains an n-c.e. (ω-c.e., respectively) set. This provides another classification of degrees below $\mathbf{0}'$, which is now known as Ershov hierarchy. As a 2-c.e. set is also the difference of two c.e. sets, we also call a 2-c.e. set a d.c.e. set and a 2-c.e. degree a d.c.e. degree. Let $\mathbf{D}_n$ be the class of all n-c.e. degrees. In this thesis, Chapters 3, 4 and 5 are devoted to the investigation on the structural interactions between different levels in the Ershov hierarchy.

We now introduce some notions of truth-table degrees. A *truth-table condition*, σ say, is a propositional formula built from atomic formulas like "$m \in X$" where X is regarded as a variable. Fix a coding scheme, such as Gödel numbering, we can computably enumerate all the truth-table conditions as follows:

$$\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots.$$

Given a set B and a truth-table condition σ, we say that B *satisfies* σ, denoted by

$B \models \sigma$, if the formula σ is true when the free variable X is interpreted as B. We say that A is *truth-table reducible, or tt-reducible for short,* to B, denoted by $A \leq_{tt} B$, if there is a total computable function f such that $x \in A$ iff $B \models \sigma_{f(x)}$. In such a case, we say that A is tt-reducible to B via f.

In general, given a partial computable function φ_e, if $\varphi_e(x)\downarrow$ then we use $[e](x)$ to denote the truth-table condition $\sigma_{\varphi_e(x)}$ and $|[e](x)|$ to denote the length of this truth-table. For any set A, $[e]^A(x)$ is defined as 0 if $A \not\models [e](x)$, and as 1 if $A \models [e](x)$. For convenience, we write $A = [e]^B$ to denote that A is tt-reducible to B via φ_e. Accordingly, A is *tt-equivalent to* B, denoted by $A \equiv_{tt} B$, iff $A \leq_{tt} B$ and $B \leq_{tt} A$, and for a given set A, the equivalence class $\mathbf{a}_{tt} = \{B \subseteq \omega : A \equiv_{tt} B\}$ is called the *tt-degree* of A, which is denoted by $deg_{tt}(A)$. Let $\mathbf{D}_{tt}$ be the class of all tt-degrees, with the partial ordering $\leq$ on $\mathbf{D}_{tt}$, the one induced by the tt-reduction. Let $\mathbf{R}_{tt}$ be the class of c.e. tt-degrees.

Chapter 2

JOINING TO HIGH DEGREES VIA NONCUPPABLES

2.1 Introduction

In this chapter, we investigate the cupping property and high/low hierarchy in **R**. To look for a natural definable ideal in **R**, Bickford and Mills [7] introduced the notion of deep degree, where a c.e. degree **a** is *deep* if $(\mathbf{a} \cup \mathbf{b})' = \mathbf{b}'$ for any c.e. degree **b**. It is clear that the deep degrees form an ideal of **R**. However, this ideal is trivial as Lempp and Slaman [50] showed that there is no nonzero deep degree. By salvaging this idea, Cholak, Groszek and Slaman [11] introduced the notion of almost deep degrees, where a c.e. degree is almost deep if it cups each low c.e. degree to a low degree, and proved the existence of nonzero almost deep degrees. Therefore the almost deep degrees form a nontrivial ideal, as they expected. In the same paper, Cholake, et al. observed that every nonzero c.e. degree can cup a low_2 c.e. degree to a nonlow_2 degree, and that for each nonzero c.e. degree **a**, there is a nonhigh c.e. degree **b** such that **b** cups **a** to a high degree. In [42], Jockusch, Li and Yang unified these two observations by showing that every nonzero c.e. degree **c** is cuppable to a high c.e. degree by a low_2 c.e. degree **b**. It is natural to ask in which subclass of low_2 c.e. degrees the **b** in [42] can be located. In [89], Wu proved that such **b** can be cappable. In the following, we will prove that such **b** can be noncuppable.

Theorem 1 (Liu and Wu [56]) *Given an incomputable c.e. degree* $\mathbf{c}$*, there is a noncuppable low*$_2$ *degree* $\mathbf{b}$ *such that* $\mathbf{c} \vee \mathbf{b}$ *is high.*

We note that in [42], Jockusch, Li and Yang proved that if $\mathbf{c}$ is a nonzero cappable degree, then there is a low$_2$ degree $\mathbf{b}$ such that $\mathbf{c} \vee \mathbf{b}$ is high, and if $\mathbf{c}$ is a noncappable degree, then it cups a low c.e. degree to $\mathbf{0}'$, by a well-known fact that a c.e. degree is noncappable if and only if it is low-cuppable (see Soare [83], Chapter XIII). Our Theorem 1 says that no matter what $\mathbf{c}$ is, if it is incomplete, then $\mathbf{c} \vee \mathbf{b}$ is always incomplete. Obviously, Theorem 1 improves both results in [42] and [89].

2.2 Requirements and basic strategies

Let $\mathbf{c} > \mathbf{0}$ be a given cappable degree and C be a fixed c.e. set in $\mathbf{c}$. We will construct a low$_2$ noncuppable set B, a c.e. set F (an auxiliary set) and partial computable functionals Γ and Δ_e for all $e \in \omega$ satisfying the following requirements:

$\mathcal{N}_e$: $\Phi_e^{B,W_e} = K \oplus F \Longrightarrow \exists \Delta_e (K = \Delta_e^{W_e})$,

$\mathcal{P}_e$: $Tot^B(e) = \lim_{x \to \infty} \Gamma^{B,C}(e, x)$,

where $\{(\Phi_e, W_e) : e \in \omega\}$ is an effective enumeration of $\{(\Phi_i, W_j) : i, j \in \omega\}$, where Φ_i is a partial computable functional and W_j is a c.e. set. $Tot^B = \{e : \Phi_e^B \text{ is total}\}$ is a Π_2^B-complete set.

By the $\mathcal{N}$ requirements, B is noncuppable. By the $\mathcal{P}$ requirements and Shoenfield Limit Lemma,

$$Tot^B \leq_T (B \oplus C)' \leq_T \emptyset''.$$

Therefore, $B'' \equiv_T (B \oplus C)' \equiv_T \emptyset''$, B is low$_2$ and $B \oplus C$ is high.

2.2.1 An $\mathcal{N}$ strategy

An $\mathcal{N}_e$-strategy, β say, is devoted to the construction of a partial computable functional Δ_β such that if $\Phi_e^{B,W_e} = K \oplus F$, then $K = \Delta_\beta^{W_e}$. As usual, we have the length of agreement functions between Φ_e^{B,W_e} and $K \oplus F$ as follows:

- $l(\beta, s) = \max\{x < s : \forall y < x[\Phi_e^{B,W_e}(y)[s] \downarrow = (K \oplus F)(y)[s]]\}$,
- $m(\beta, s) = \max\{l(\beta, t) : t < s$ and t is a β-stage$\}$.

Here say that a stage s is β-stage if β is visited at stage s. Say a stage s is a β-expansionary if $s = 0$ or s is a β-stage with $l(\beta, s) > m(\beta, s)$. We only define Δ_β at the β-expansionary stages. That is, if s is β-expansionary, and $\Delta_\beta^{W_e}(x)[s]$ is not defined with $2x + 1 < l(\beta, s)$, then define $\Delta_\beta^{W_e}(x) = K(x)[s]$ with use s, which is bigger than $\varphi_e(2x)[s]$ and $\varphi_e(2x + 1)[s]$, and after stage s, $\Delta_\beta^{W_e}(x)$ can be undefined only when W_e changes below $\varphi_e(2x + 1)$. So in case that x is enumerated into K, then we should force W_e to change below $\varphi_e(2x + 1)$ (otherwise, $K \oplus F$ and Φ_e^{B,W_e} will differ at $2x+1$, and $\mathcal{N}$ is satisfied vacuously), so that we can redefine $\Delta_\beta^{W_e}(x)$ as $K(x)$ afterwards. β has two possible outcomes: an infinitary outcome i and a finitary outcome f.

Problems arise when numbers are enumerated into B by the $\mathcal{P}$-strategies (to make $B \vee C$ high) below the infinitary outcome i of β. Suppose that at stage s, $\Delta_\beta^{W_e}(x)[s]$ is defined and a number n less than $\varphi_e(2x)[s]$ is enumerated into B, and such an enumeration can lift the use $\varphi_e(2x)$ up to a bigger number. Now suppose that x enters K, before the next β-expansionary stage s' say. Then

$$\Phi_e^{B,W_e}(2x)[s'] = 1 = (K \oplus F)(2x)[s'].$$

However, as W_e has no changes below $\varphi_e(2x)[s]$, $\Delta_\beta^{W_e}(x)[s']$ is defined as $\Delta_\beta^{W_e}(x)[s]$, and is equal to 0, so $\Delta_\beta^{W_e}$ is not correct at x.

To avoid such a scenario, when we want to put a number, n say, into B, at stage s, where n is selected at stage $s_0 < s$, we want to ensure that no $\Delta_\beta^{W_e}(x)$ defined at stage s is actually defined after stage s_0. Thus, if $\Delta_\beta^{W_e}(x)[s]$ has definition at stage s, then it is defined before stage s_0, and the enumeration of n into B will not change the computation $\Phi_e^{B,W_e}(2x)$, as n is bigger than the corresponding use.

Now we explain the idea of ensuring that all $\Delta_\beta^{W_e}(x)$ having definition at stage s are actually defined before stage s_0. When we choose n at stage s_0, we also choose an auxiliary number, a_β^n big, and we think that the next β-expansionary stage, t say, should be a stage with $l(\beta, t) > m(\beta, t)$ and also $l(\beta, t) > 2a_\beta^n + 1$ (we extend the definition of Δ_β at this stage). In this way, the construction is delayed a little bit, but if there are infinitely many β-expansionary stages, then it makes no difference.

Assume that $\Delta_\beta^{W_e}(x)$ is defined at stage t, but before stage s (remember that at stage s, we want to put n into B), then we first put a_β^n into F. There are two cases. The first case is that W_e does not change below $\varphi_e(2a_\beta^n+1)$, then $\Phi_e^{B,W_e}(2a_\beta^n+1) = 0$ and $(K \oplus F)(2a_\beta^n+1) = 1$, and β is satisfied vacuously, and of course, there are no more β-expansionary stages. The other case is that a new β-expansionary stage appears, which means that W_e does change below $\varphi_e(2a_\beta^n + 1)$. This W_e-change undefines $\Delta_\beta^{W_e}(x)$, which are defined after stage t. As no $\Delta_\beta^{W_e}(x)$ is defined between stages s_0 and t, at stage $s' > s$, the next expansionary stage, no $\Delta_\beta^{W_e}(x)$ is defined between stages s_0 and s', and as pointed out above, we can now put n into B as wanted. Definitely, this enumeration keeps the definition of $\Delta_\beta^{W_e}(x)$ correct.

We now consider a general situation. On the priority tree, when we want to put a number, a γ-use, into B at a $\mathcal{P}$-strategy, α say, we need to make sure that α works consistently with all the $\mathcal{N}$-strategies with priority higher than α. Without loss of generality, we assume that $\beta_0, \beta_1, \cdots, \beta_n$, with $\beta_0{}^\frown i \subseteq \beta_1{}^\frown i \subseteq \cdots \subseteq \beta_n{}^\frown i \subseteq \alpha$, are the $\mathcal{N}$-strategies with priority higher than α and outcome i. Then, when α defines a γ-use, at stage s_0, say, it also defines a sequence of auxiliary numbers $z_0, z_1, \cdots, z_n$ (actually, we will select more such numbers, because the $\mathcal{P}$-strategies to the left of α will be also considered, as one of these strategies can be the one enumerating this γ-use into B). Now, for each $i \leq n$, say that a stage s is β_i-expansionary only when the length of agreement between $\Phi_{\beta_i}^{B,W_{\beta_i}}$ and $K \oplus F$ is greater than $2z_i + 1$. When α wants to put $\gamma(e(\alpha), y)$ into B, α puts z_n into F first, and waits for the next β_n-expansionary stage. For convenience, we create a link between α and β_n, and at the next β_n-expansionary stage, we go to α via this link, and do further actions. If there is no such a β_n expansionary stage, then β_n has finitary outcome, and we do not need to satisfy α at all. Otherwise, at the next β_n-expansionary stage, we cancel the link between α and β_n, put z_{n-1} into F, and create a link between α and β_{n-1}. Again, we wait for the next β_{n-1}-expansionary stage, and so on. Such a process can be iterated at most $n+1$ many times, each of which will have corresponding W-changes on some small numbers, undefining those $\Delta_{\beta_i}^{W_{\beta_i}}$ defined after stage s_0. So if eventually we cancel the link between α and β_0, we have actually forced that all of the $\Delta_{\beta_i}^{W_{\beta_i}}$ defined after stage s_0 are undefined, and now we can enumerate $\gamma(e(\alpha), y)$ into B, and this enumeration is consistent with all the $\mathcal{N}$-strategies with higher priority.

2.2.2 A $\mathcal{P}$ strategy

The basic idea of a $\mathcal{P}_e$ strategy is to approximate $Tot^B(e)$ via $\Gamma^{B,C}(e, -)$ in the limit, where Γ is a (global) partial computable functional defined throughout the whole construction. With this in mind, we need to ensure that $\Gamma^{B,C}$ is totally defined, and that for all $e \in \omega$, $\lim_{x\to\infty} \Gamma^{B,C}(e, x)$ exists, with

$$\Phi_e^B \text{ is total iff } \lim_{x\to\infty} \Gamma^{B,C}(e, x) = 1.$$

We first consider a single $\mathcal{P}$ strategy. This is a modified version of the gap-cogap argument developed in [42] by Jockusch, Li and Yang.

Let α be a $\mathcal{P}_e$ strategy. Assume that $|\alpha| = e$. For convenience, we write Φ_α for Φ_e. α will do two jobs simultaneously. α's first job is to define $\Gamma^{B,C}(e, x)$ for almost all x to make sure that $\Gamma^{B,C}(e, x)$ has a limit and computes $Tot^B(e)$ correctly. That is, the following equality is guaranteed:

$$Tot^B(e) = \lim_{x\to\infty} \Gamma^{B,C}(e, x).$$

In the construction, whenever α defines $\Gamma^{B,C}(e, x)$ at stage s, the γ-use $\gamma(e, x)$ is defined as a big number. In particular, $\gamma(e, x) > s$. $\Gamma^{B,C}(e, x)$ is undefined automatically if some number less than or equal to $\gamma(e, x)$ is enumerated into B. As a consequence, $\gamma(e, x)$ may be lifted when $\Gamma^{B,C}(e, x)$ is redefined later. If C has a change below $\gamma(e, x)$ (it can happen when we get a cogap permission as specified later), unless we explicitly set $\Gamma^{B,C}(e, x)$ to be undefined, $\Gamma^{B,C}(e, x)$ is redefined with same value and the same use (it is not necessary to lift $\gamma(e, x)$ to a big number) to ensure that $\Gamma^{B,C}(e, x)$ is defined eventually. Such rules also apply to the Θ_α-functionals defined later.

As in our construction a $\mathcal{P}_e$ strategy needs to be consistent with the $\mathcal{N}$-strategies with higher priority, when we define $\gamma(e,x)$ we also select several other associated big numbers, z_j say. We do so, because when we want to enumerate $\gamma(e,x)$ into B, we will put these numbers into F one by one, as described in the $\mathcal{N}$-strategy, to force changes of W to undefine Δ_β^W defined after stage s.

α's second job is to preserve $\Phi_e^B(x)$, to ensure that if $\Gamma^{B,C}(e,x)$ has limit 1, then Φ_e^B is total. At stage s, we define the length of convergence functions as follows:

- $l(\alpha,s) = \max\{x < s : \Phi_\alpha^B(y)[s]\downarrow \text{ for all } y < x\}$,
- $m(\alpha,s) = \max\{0, l(\alpha,t) : t < s \text{ and } t \text{ is a } \alpha\text{-stage}\}$.

Say that s is an α-expansionary stage if $s = 0$ or $m(\alpha,s) > l(\alpha,s)$.

If there are only finitely many α-expansionary stages, then Φ_α^B is obviously not total. Thus, $Tot^B(e) = 0$, and α will define $\Gamma^{B,C}(e,x) = 0$ for (almost) all $x \in \omega$, and eventually, we have that $\lim_{x\to\infty}\Gamma^{B,C}(e,x) = 0 = Tot^B(e)$.

Suppose that there are infinitely many α-expansionary stages. Then we should ensure that Φ_α^B is total and the values $\Gamma^{B,C}(e,x)$ are defined to be 1 for (almost) all $x \in \omega$ such that $Tot^B(e) = \lim_{x\to\infty}\Gamma^{B,C}(e,x) = 1$. Here comes a direct conflict: to preserve a computation $\Phi_\alpha^B(x)$, we need to preserve B on the use $\varphi_\alpha(x)$, and to change $\Gamma^{B,C}(e,x)$ from 0 (defined by a strategy to the right) to 1, we may need to enumerate $\gamma(e,x)$ into B to undefine $\Gamma^{B,C}(e,x)$ first. Fortunately, it is not a fatal conflict, as we can use the C-changes to undefine $\Gamma^{B,C}(e,x)$. With this in mind, to preserve a computation $\Phi_\alpha^B(x)$, we introduce the following substrategies, $\mathcal{S}_{\alpha,i}, i \in \omega$, of α, to undefine $\Gamma^{B,C}(e,x)$, and to preserve $\Phi_\alpha^B(x)$, if needed.

For the sake of the consistency between defining $\Gamma^{B,C}$ and preserving $\Phi_\alpha^B(x)$, $x \in \omega$, α will construct an auxiliary c.e. set E_α and a partial functional $\Theta_\alpha^{E_\alpha,C}$, which attempts to satisfy the following requirements:

$$\mathcal{S}_{\alpha,i}: \ E'_\alpha(i) = \Theta_\alpha^{E_\alpha,C}(i).$$

An $\mathcal{S}_{\alpha,i}$-strategy works at α-expansionary stages. It defines $\Theta_\alpha^{E_\alpha,C}(i)$ with $\Theta_\alpha^{E_\alpha,C}(i) = E'_\alpha(i)$, and if it fails, then it will ensure that Φ_α^B is total and $\lim_{x\to\infty}\Gamma^{B,C}(e,x) = 1$, satisfying $\mathcal{P}_e$. As C is given as a set with cappable degree, not every $\mathcal{S}_{\alpha,i}$ can be satisfied, as otherwise, E_α would be a low set, and C is low-cuppable, which is a contradiction. Therefore, there is a least i such that $\mathcal{S}_{\alpha,i}$ is not satisfied.

Fix i. Say that *$\mathcal{S}_{\alpha,i}$ is in the x-turn (or x-turn is in progress) if $\mathcal{S}_{\alpha,i}$ attempts to make $\Phi_\alpha^B(x)$ clear of the $\gamma(e,-)$ uses — no further enumerations of $\gamma(e,-)$ uses can change the computation $\Phi_\alpha^B(x)$.* $\mathcal{S}_{\alpha,i}$ works as follows:

1. Suppose that the x-turn is in progress.

   Wait for an α-expansionary stage $s_1 > 0$ with $l(\alpha, s_1) > x$. If $\Theta_\alpha^{E_\alpha,C}(i)$ is undefined at stage s_1, then define $\Theta_\alpha^{E_\alpha,C}(i)[s_1]$ as 0 if $\Phi_i^{E_\alpha}(i)[s_1]\uparrow$, and 1 otherwise. In both cases, set the use $\theta_\alpha(E_\alpha, C; i)[s_1]$ big.

2. Wait for an α-expansionary stage $s_2 > s_1$ with $\Phi_i^{E_\alpha}(i)[s_2]\downarrow$ and $\Theta_\alpha^{E_\alpha,C}(i)[s_2]\downarrow= 0$. We say that we are ready to open a gap at stage s_2, and we will put those numbers associated to $\gamma(e,x)$ into F one by one, as described in the $\mathcal{N}$-strategy.

   When $\gamma(e,x)$ is defined by a strategy α', for the sake of the consistency between the $\mathcal{P}$-strategies and the $\mathcal{N}$-strategies, we also select several numbers $z_0^\xi, z_1^\xi, \cdots, z_e^\xi$, where ξ is a $\mathcal{S}_{\beta,j}$-strategy to the left of α', in case when $\gamma(e,x)$ is enumerated into B by ξ, ξ needs to enumerate the associated numbers $z_0^\xi, z_1^\xi, \cdots,$ z_e^ξ into F to force the corresponding Ws to have the needed changes. As there are only finitely many such ξ-strategies to the left of α', it is fine for us to select these auxiliary numbers.

   When α wants to put $\gamma(e,x)$ into B, α, similarly to what is done in the $\mathcal{N}$-strategies, will put numbers $z_0^\alpha, z_1^\alpha, \cdots, z_e^\alpha$ into F in reverse order, creating and canceling the corresponding links. After the last link is canceled, α puts $\gamma(e,x)$ into B. This delays the opening of a gap — we open a gap only when $\gamma(e,x)$ is put into B, because before $\gamma(e,x)$ is put into B, a C-change below $\gamma(e,x)$ can always undefine $\Gamma^{B,C}(e,x)$, and we can protect $\Phi_\alpha^B(x)$ successfully. Obviously, this delay does not affect the $\mathcal{S}_{\alpha,i}$-strategy. So at a stage $s_3 > s_2$ at which the last link between α and the highest $\mathcal{N}$-strategy above α is canceled, we open a gap. This is a new, crucial feature of our construction.

It may happen that α' is to the left of α, then we will not allow α to enumerate $\gamma(e,x)$ into B, as α has lower priority. If α is on the true path, then only finitely many $\mathcal{S}$-strategies to the left can be accessible during the whole construction, and as a consequence, there are only finitely many x with $\Gamma^{B,C}(e,x)$ defined by these higher priority strategies. This will not affect the limit $\lim_{x\to\infty} \Gamma^{B,C}(e,x)$, which is in the control of α.

3. Let $s_3 > s_2$ be a stage at which the last link between α and the highest $\mathcal{N}$-strategy above α is canceled. Open a gap as follows:

   - Set $r(\alpha^\frown g_i) = 0$, where g_i is an gap-outcome of α, saying that $\mathcal{S}_{\alpha,i}$ opens infinitely many gaps (and hence closed unsuccessfully) in the construction. Define $f^x_{\alpha,i}(y) = C_{s_3}(y)$ for those $y < \theta_\alpha(E_\alpha, C; i)[s_2]$ with $f^x_{\alpha,i}(y)$ not defined yet.

     (*Here, $f^x_{\alpha,i}$ is an auxiliary partial computable function defined during the x-turn, threatening to make C computable if the x-turn does not terminate.*)
   - Enumerate $\gamma(e,x)$ into B.

4. Wait for the least α-expansionary stage $s_4 > s_3$. α closes the gap which is opened at stage s_3. There are two cases, depending on whether C has a wanted change.

   - (Successful close) C has a change below $\theta_\alpha(E_\alpha, C; i)[s_2]+1$ between stages s_2 and s_4. Then redefine $\Theta^{E_\alpha,C}_\alpha(i) = 1$. Note that this C-change undefines all $\Theta^{E_\alpha,C}_\alpha(j)$, $j \geq i$. Declare that the gap is closed successfully and that $\mathcal{S}_{\alpha,i}$ is satisfied. ($\mathcal{S}_{\alpha,i}$ *succeeds in defining* $\Theta^{E_\alpha,C}_\alpha(i) = E'_\alpha(i)$ *because the computation* $\Phi^{E_\alpha}_i(i)[s_2]$ *is preserved from now on and hence* $\Phi^{E_\alpha}_i(i)\downarrow$.)
   - (Unsuccessful close) C has no change below $\theta_\alpha(E_\alpha, C; i)[s_2] + 1$ between stages s_2 and s_4. Then define $r(\alpha^\frown g_i) = s_4$ and enumerate $\theta_\alpha(E_\alpha, C; i)[s_2]$ into E_α. This enumeration undefines all $\Theta^{E_\alpha,C}_\alpha(j)$, $j \geq i$. Declare that the gap is closed unsuccessfully. Go to (1) and simultaneously, wait for a

C-change on a number in $\text{dom}(f^x_{\alpha,i})$, till the stage when the next gap is open. If C has such a change on $\text{dom}(f^x_{\alpha,i})$, go to (5).

(Here, we set $r(\alpha^\frown g_i) = s_4$ as a restraint to preserve the computation $\Phi^B_e(x)$ to keep the same value as $\Phi^B_e(x)[s_4]$ until this restraint is canceled in (3), when a new gap is open, in which case, $\Phi^B_e(x)$ may be injured by the enumeration of $\gamma(e,x)$, or when (5) is reached. If before we open a new gap, C does have a change below $\theta_\alpha(E_\alpha, C; i)[s_2]+1$, as this C change lifts the γ-uses to big numbers, $\mathcal{S}_{\alpha,i}$ succeeds in preserving the computation $\Phi^B_e(x)$ to keep the same value as $\Phi^B_e(x)[s_4]$.)

In any case, find the least y with $\Gamma^{B,C}(e,y)$ undefined, and define $\Gamma^{B,C}(e,y) = 1$ with use $\gamma(e,y)$ big.

5. Define $r(\alpha^\frown g_i) = 0$. Declare that the computation $\Phi^B_e(x)$ is preserved by $\mathcal{S}_{\alpha,i}$. Terminate the x-turn (and hence stop defining $f^x_{\alpha,i}$, as it is not correct anymore), and start the $x+1$-turn (to preserve $\Phi^B_e(x+1)$). We call such a C-change a cogap permission, because it happens inside a cogap. The action performed at (5) is called a cogap permission action.

Say that $\mathcal{S}_{\alpha,i}$ requires attention at an α-expansionary stage s if one of the following holds:

1. $\mathcal{S}_{\alpha,i}$ is inside a gap, and it is ready to close a gap (at step 4). α will act by closing this gap, no matter whether it is a successful close or not.
2. $\mathcal{S}_{\alpha,i}$ is inside a cogap and $\Theta^{E_\alpha,C}_\alpha(i)[s]\uparrow$ (we will define $\Theta^{E_\alpha,C}_\alpha(i)$). α will act by defining $\Theta^{E_\alpha,C}_\alpha(i) = 1$ if $\Phi^{E_\alpha}_i(i)[s]\downarrow$, or 0 if $\Phi^{E_\alpha}_i(i)[s]\uparrow$.
3. $\mathcal{S}_{\alpha,i}$ is inside a cogap and is ready to open a gap (at step 2, and will put the associated numbers into F one by one). α will act by enumerating the numbers $z^\alpha_j, j \le e$ into F one by one, as described in the $\mathcal{N}$-strategy.
4. $\mathcal{S}_{\alpha,i}$ is at a stage at which a last link between α and an $\mathcal{N}$-strategy above α with the highest priority is canceled (at step 3, and will open a new gap). α will act by enumerating $\gamma(e,x)$ into B and open a gap.

5. $\mathcal{S}_{\alpha,i}$ is at a cogap and C changes below $f^x_{\alpha,i}$ (step 5 is reached). We will start the $x+1$-turn.

$\mathcal{S}_{\alpha,i}$ has three possible outcomes:

s_i: $\mathcal{S}_{\alpha,i}$ waits at step 1 or closes a gap successfully. In the latter case, $\mathcal{S}_{\alpha,i}$ succeeds in defining $\Theta^{E_\alpha,C}_\alpha(i) = E'_\alpha(i)$. $\mathcal{S}_{\alpha,i}$ is satisfied. We will consider the definition of $\Theta^{E_\alpha,C}_\alpha(i+1)$, so we will not put any outcome on the tree of strategies. In the former case, there are only finitely many α-expansionary stages and α is satisfied, because Φ^B_α is not total. In this case, we have an outcome f, under which $\Gamma^{B,C}(e,x)$ will be defined as 0 for almost all x. $\mathcal{S}_{\alpha,i}$ is again satisfied, because $\lim_{x\to\infty}\Gamma^{B,C}(e,x) = 0$, and equals to $Tot^B(e)$.

g_i: α opens infinitely many gaps and closes these gaps unsuccessfully, and for each $x \in \omega$, the x-turn stops by reaching step 5 (a wanted C-change appears). As a consequence, $\mathcal{S}_{\alpha,i}$ ensures that $\Phi^B_e(x)$ converges. So Φ^B_e is total. As we always define $\Gamma^{B,C}(e,x) = 1$ for $x \in \omega$ when we open a gap at step 3, $\lim_{x\to\infty}\Gamma^{B,C}(e,x) = 1$. Thus, $Tot^B(e) = \lim_{x\to\infty}\Gamma^{B,C}(e,x)$, and so $\mathcal{P}_e$ is satisfied;

f_i: After some sufficiently large stage, $\mathcal{S}_{\alpha,i}$ has an x-turn opening and closing (unsuccessfully) infinitely many gaps and never reaching step 5. Then $\Theta^{E_\alpha,C}_\alpha(i)$, and $\Phi^B_\alpha(x)$ diverge, but $f^x_{\alpha,i}$ is totally defined and computes C correctly because C does not change in a gap (otherwise we can undefine $\Theta^{E_\alpha,C}_\alpha(i)$ and preserve $\Phi^B_\alpha(x)$) nor in a cogap (otherwise, the x-turn is terminated and the $x+1$-turn would be started). So C is computable, contradicting our assumption on C. As f_i will not happen, we do not put this outcome on the priority tree.

In the construction, we do not put the $\mathcal{S}_{\alpha,i}$-substrategy on the priority tree. We just attach the outcome of $\mathcal{S}_{\alpha,i}$ to α. As explained above, for each $i \in \omega$, $\mathcal{S}_{\alpha,i}$ has exactly one outcome g_i listed on the priority tree. As a consequence, α has $\omega+1$ many outcomes,

$$g_0 <_L g_1 <_L \cdots <_L g_i <_L \cdots <_L f,$$

where f denotes the case that there only finitely many α-expansionary stages and g_i denotes the case that $\mathcal{S}_{\alpha,i}$ requires attention (and hence receives attention) infinitely often.

Now, we consider how to make two (and more) $\mathcal{P}$-strategies consistent. Note that at step 5 of the $\mathcal{S}_{\alpha,i}$-strategy, α successfully ensures that the computation $\Phi_\alpha^B(x)$ is clear of the $\gamma(e,-)$ uses, and sets $r(\alpha^\frown g_i) = 0$ when step 5 is reached. It may happen that a computation $\Phi_\alpha^B(x)$ can still be changed by another $\mathcal{P}$-strategy (via enumeration of other γ-uses) below outcome g_i. We will explain how we can deal with such disturbance coming from lower priority strategies. In the construction, we will ensure that for any x, $\Phi_\alpha^B(x)$ can be changed by the strategies below outcome g_i at most finitely often, and finally if we have a new α-expansionary stage, at which $\Phi_\alpha^B(x)$ converges, then again, no strategy can change this computation and hence it is preserved forever.

Here is the point. Let α, α' be two $\mathcal{P}$-strategies with $\alpha^\frown g_i \subseteq \alpha'$. For the sake of simplicity, in the remainder of this section, when we talk about $\gamma(e,-)$, we mean $\gamma(e(\alpha),-)$. Supposing that α' wants to define $\Gamma^{B,C}(e(\alpha'),y)$ for some y, we first set k^y as a big number, and wait for a stage, s say, such that $\mathcal{S}_{\alpha,i}$ has completed the k^y-turn, and then define $\Gamma^{B,C}(e(\alpha'),y)$ at this stage. That is, at stage s, all computations, $\Phi_\alpha^B(z)$, $z < k^y$, are clear of the $\gamma(e(\alpha),-)$-uses. Again, it delays the definition of $\Gamma^{B,C}(e(\alpha'),y)$. However, we know that such an s exists because α' guesses that α has outcome g_i, and hence Φ_α^B is total. We will call k^y an α-bound at y. Obviously, $\gamma(e(\alpha'),y)$ is defined as a number bigger than s, and hence if it is enumerated into B, it will not change the computations $\Phi_\alpha^B(z)$, $z < k^y$. So for a fixed $x < k^y$, after stage s, $\Phi_\alpha^B(x)$ can be changed by α' at most finitely often by the enumeration of those $\gamma(e(\alpha'),y')$, $y' < y$, and if there are infinitely many α-expansionary stages, then we will finally have a computation $\Phi_\alpha^B(x)$ not injured by α'. Also note that by stage s, as the x-turn has already been completed, $\Phi_\alpha^B(x)$ is clear of the $\gamma(e,-)$-uses, so whenever α' enumerates a number into B, changing a computation $\Phi_\alpha^B(x)$, the corresponding $\gamma(e,-)$-uses are also lifted up to even bigger numbers, and once $\Phi_\alpha^B(x)$ settles down, we can make sure to keep these $\gamma(e,-)$-uses to be the same. Therefore, α' is consistent with α.

Now we consider how α' can work below $\alpha^\frown g_i$, where α' knows that α will open infinitely many gaps and hence put infinitely numbers into B. As usual, when α' sees a computation, α' first checks whether all $\gamma(e,z)$s (for $\Gamma^{B,C}(e,z) = 0$) below the associated use of this computation have been enumerated into B. If yes, then α' knows that this computation will not be changed by the enumeration performed by α, and will preserve this computation, as described above in the $\mathcal{N}$ and $\mathcal{P}$-strategies. In this case, we say that this computation is believable at α'. Otherwise, because α' is below the outcome g_i of α, and α' knows that all these $\gamma(e,-)$-uses will be put into B sooner or later, α' will just wait until all these uses are bigger than the associated use — wait till a computation is believable at α'. Again, it is a kind of delaying, and such a delay will not affect the whole construction at all.

Now, for a fixed y, we define $\gamma(e(\alpha'),y)$ only when $\Phi^B_{\alpha'}(y)$ is a believable computation at α'. So α's enumeration will not affect the definition of $\Phi^B_{\alpha'}(y)$, and we can argue as usual that eventually, after all these $\gamma(e(\alpha'),y')$ with $y' < y$ become fixed, $\gamma(e(\alpha'),y)$ can be changed only when the corresponding substrategy of α' at step 3, or by a C-change at step 5. As C is assumed to be incomputable, step 5 is eventually reached, after which we will not change $\gamma(e(\alpha'),y)$ anymore.

We are now ready to give the full construction.

2.3 Construction

First, we define the priority tree T, and assign requirements to the nodes on T. Given $\sigma \in T$.

- If $|\sigma| = 2e$, then σ is an $\mathcal{N}$-strategy, and has outcomes i and f with $i <_L f$.
- If $|\sigma| = 2e+1$, then σ is a $\mathcal{P}_e$-strategy, and has outcomes $g_0, g_1, \cdots, g_n, \cdots, f$ with $g_0 <_L g_1 <_L g_2 <_L \cdots <_L g_n <_L \cdots <_L f$.

In the following, for a node σ, we refer to $e(\sigma)$ as the index of the $\mathcal{N}$-strategy or the $\mathcal{P}$-strategy to which σ is devoted. In the following, the computation $\Phi^B_\sigma(x)[s]$ is always σ-believable, i.e., for any $\mathcal{P}$-strategy α with $\alpha^\frown g_i \subseteq \sigma$ for some i, the use of the computation $\Phi^B_\sigma(x)[s]$, $\varphi_\sigma(x)[s]$, is less than $\gamma(e(\alpha), y)$ for any y with $\Gamma^{B,C}(e(\alpha), y)[s] = 0$. Thus the length of agreement functions are defined as follows: for any $\sigma \in T$,

- If $\sigma = \beta$ is an $\mathcal{N}$-strategy, define

  $l(\beta, s) = \max\{x : (\forall y < x)(\Phi^{B,W_e}_\beta(y)[s] \downarrow = (K \oplus F)(y)[s]$ and $\Phi^{B,W_e}_\beta(y)[s]$ is β-believable$)\}$,

  $m(\beta, s) = \max\{0, l(\beta, t) : t < s$ and t is a β-stage$\}$.

  Say that s is a β-expansionary stage if $s = 0$ or $l(\beta, s) > m(\beta, s)$.

- If $\sigma = \alpha$ is a $\mathcal{P}$-strategy, define

  $$l(\alpha, s) = \max\{x < s : \text{for all } y < x, \Phi^B_\alpha(y)[s] \downarrow \text{ is } \alpha\text{-believable}\}$$
  $$m(\alpha, s) = \max\{0, l(\alpha, t) : t < s \text{ and } t \text{ is a } \alpha\text{-stage}\}.$$

  Say that s is an α-expansionary stage if $s = 0$ or $l(\alpha, s) > m(\alpha, s)$.

Now we define the restraints on the tree. For σ on T, $r(\sigma)[s]$ is the restraint imposed by σ at stage s and $R(\sigma)[s]$ is the restraint imposed by σ itself and also those strategies with higher priority valid at stage s. When a node σ is initialized, then all the sets, the parameters (except those γ-uses) and the functionals (except Γ) associated to are reset to be completely undefined. In particular, if σ is an $\mathcal{N}$-strategy, then when it is initialized, Δ_σ becomes undefined completely, and all the associated links are canceled, and if σ is an $\mathcal{P}$-strategy, then when it is initialized, E_σ and Θ_σ becomes undefined completely. In the latter case, all the substrategies of σ are initialized, and all the associated links are canceled.

Full Construction

In the following, we give the full construction. At the end of each stage s, define σ_s as the current approximation of the true path. Say that s is a ξ-stage, if $\xi \subseteq \sigma_s$. Without loss of generality, we assume that C is enumerated only at odd stages.

Stage 0. Initialize all nodes first. Let B be $\emptyset$, and $\Gamma^{B,C}$ be totally undefined.

Stage $s = 2e$ ($e > 0$). We inductively define a partial function σ_s of length $\leq s$ by substages. First, let $\sigma_{s,0} = \lambda$, the root node.

Substage t. Given $\tau = \sigma_{s,t}$. Initialize all nodes to the right of τ. If $t = s$, then define $\sigma_s = \tau$, and go to the next stage. Otherwise, take actions for τ, and determine τ's outcome as follows.

Case 1. $\tau = \beta$ is an $\mathcal{N}$-strategy.

There are two subcases:

$\beta 1$. s is not β-expansionary.

Then f is β's current outcome, and we let $\sigma_{s,t+1} = \beta^\frown f$, i.e. let $\beta^\frown f$ be eligible to act at the next substage. *Note that in this case, we do not extend the definition of $\Delta_\beta^{W_\beta}$.*

$\beta 2$. s is β-expansionary.

In this case, we first see whether there is a link between β and some $\mathcal{P}$-strategy $\alpha \supseteq \beta^\frown i$.

- If there is such a link, then cancel this link and see whether there is an $\mathcal{N}$-strategy β' with $\beta'^\frown i \subseteq \beta$. If there is such a β' strategy, then create a link between α and such a β' with the lowest priority, and stop stage s. Define $\sigma_s = \sigma_{s'}$ where $s' < s$ is the stage at which this cancelled link was created. If there is no such a β', then α opens a gap at this stage by letting $r(\alpha^\frown g_i) = 0$ (we assume that α is running substrategy $\mathcal{S}_{\alpha,i}$), and enumerate the associated number, $\gamma(e(\alpha), y)$ say, into B, and request $\gamma(e(\alpha), y)$ be defined big. Define $f^x_{\alpha,i}(z) = C_s(z)$ for those $z < \theta_\alpha(E_\alpha, C; i)[s]$ if $f^x_{\alpha,i}(z)$ is not defined currently. *Note that the enumeration of $\gamma(e(\alpha), y)$ into B undefines $\gamma(e(\alpha'), x)$, where $\alpha' \subset \alpha$ is a $\mathcal{P}$-strategy with $x > k^y_{\alpha'}$, and $k^y_{\alpha'}$ is the α'-bound of y.* Let $\sigma_{s,t+1} = \alpha^\frown g_i$, i.e. let $\alpha^\frown g_i$ be accessible at the next substage.

- If there is no such a link, then let $\sigma_{s,t+1} = \beta^\frown i$, i.e. let $\beta^\frown i$ be eligible to act at the next substage. Find those x such that $2x + 1 < l(\beta, s)$ and $\Delta_\beta^{W_\beta}(x)[s]$

is not defined and define $\Delta_\beta^{W_\beta}(x) = K(x)[s]$ with use s, *which is bigger than* $\varphi_\beta(2x)[s]$, $\varphi_\beta(2x+1)[s]$.

Case 2. $\tau = \alpha$ *is a* $\mathcal{P}$*-strategy.*

There are two subcases:

$\alpha 1$. s is not α-expansionary.

(1) First find the least y with $\Gamma^{B,C}(e(\alpha), y)[s]$ defined by the (sub)strategies, α' say, on the right of $\alpha^\frown f$ and $\gamma(e(\alpha), y) > R(\alpha^\frown f)[s]$.

Check whether there are $\mathcal{N}$-strategies β with $\beta^\frown i \subseteq \alpha, \alpha'$. If there are such $\mathcal{N}$-strategies, find one with lowest priority, β' say, and create a link between α and β', define $\sigma_s = \xi$, where ξ is a strategy below $\alpha^\frown f$ with lowest priority which has been accessible earlier, and has never been initialized since then. *That is, the strategies below* $\alpha^\frown f$ *are frozen from now on, and will be active again after all the links created at* $\alpha^\frown f$ *are canceled later.* Stop stage s. If there are no such $\mathcal{N}$-strategies, then enumerate $\gamma(e(\alpha), y)$ into B, and request that $\gamma(e(\alpha), y')$ with $y' \geq y$ be defined big. Let $\sigma_{s,t+1} = \alpha^\frown f$, i.e. let $\alpha^\frown f$ be accessible at the next substage.

(2) Find the least $x \leq s$ with $\Gamma^{B,C}(e(\alpha), x)$ undefined currently. (If there is no such an x, then do nothing.)

If $k^x_{\alpha',f}$, $\alpha' \subset \alpha$, are not defined, then define these numbers (called α' bounds of x) as big numbers. If these bounds are defined, then define $\Gamma^{B,C}(e(\alpha), x) = 0$, or 1, depending on whether a computation of $\Phi^B_\alpha(x)$ has been preserved. The use $\gamma(e(\alpha), x)[s]$ is selected as follows:

- If $\gamma(e(\alpha), x)[s]$ has not been defined or is requested to be big, then define it as a big number. At the same time, select a sequence of big numbers $z^x_{\xi,0}, z^x_{\xi,1}, \cdots, z^x_{\xi,e(\alpha)}$, where ξ is a $\mathcal{P}_{e(\alpha)}$-strategy with priority higher than or equal to α.
- Otherwise, the previous definition of $\Gamma^{B,C}(x)$ is undefined because of the C-change. In this case, we define $\gamma(e(\alpha), x)[s]$ as the previous one.

In both cases, let $\sigma_{s,t+1} = \alpha^\frown f$, i.e. let $\alpha^\frown f$ be accessible at the next substage.

α2. s is α-expansionary.

Let i be the least number such that substrategy $\mathcal{S}_{\alpha,i}$ requires attention. There are three subcases.

- Subcase α2.1. $\mathcal{S}_{\alpha,i}$ is inside a cogap and $\Theta_\alpha^{E_\alpha,C}(i)$ is not defined at stage s. Then let $\Theta_\alpha^{E_\alpha,C}(i)[s] = 0$ if $\Phi_i^{E_\alpha}(i)[s] \uparrow$, and 1 otherwise. Similar to the definition of the γ-uses described above, the use $\theta_\alpha(E_\alpha, C; i)[s]$ is defined as a big number if it has no definition before or it is requested to be defined as big. Otherwise, we define $\theta_\alpha(E_\alpha, C; i)[s]$ the same as the previous one (note that it is caused by a C-change). Define $\sigma_s = \alpha^\frown g_i$, and go to the next stage.

- Subcase α2.2. $\mathcal{S}_{\alpha,i}$ is inside a cogap and $\Phi_i^{E_\alpha}(i)[s] \downarrow$, $\Theta_\alpha^{E_\alpha,C}(i)[s] = 0$ ($\mathcal{S}_{\alpha,i}$ is ready to open a gap). Suppose that the x-turn is now in progress.

  Find the least y with $\Gamma^{B,C}(e(\alpha), y)[s] \downarrow$ and $\gamma(e(\alpha), y) > R(\alpha^\frown g_i)[s]$ is defined by the (sub)strategies to the right of $\alpha^\frown g_i$ (including the case that $\Gamma^{B,C}(e(\alpha), y)[s]$ is defined by $\alpha^\frown f$ as 0). (If there is no such a y then do nothing.) Check whether there are $\mathcal{N}$-strategies β with $\beta^\frown i \subseteq \alpha$.

  If there are such $\mathcal{N}$-strategies, find one with the lowest priority, β' say, and create a link between $\alpha^\frown g_i$ and β'. Define $\sigma_s = \xi$, where ξ is a strategy below $\alpha^\frown g_i$ with lowest priority which has been accessible before, and from then on, has not been initialized. *We do this as we want to freeze those strategies below $\alpha^\frown g_i$ till all necessary links are created and cancelled.* Stop stage s.

  If there are no such $\mathcal{N}$-strategies, then open a gap by letting $r(\alpha^\frown g_i) = 0$, enumerating $\gamma(e(\alpha), y)$ into B, and requesting $\gamma(e(\alpha), y)$ be defined as big. Define $f_{\alpha,i}^x(z) = C_s(z)$ for those $z < \theta_\alpha(E_\alpha, C; i)[s]$ if $f_{\alpha,i}^x(z)$ is not defined currently. Let $\sigma_{s,t+1} = \alpha^\frown g_i$, i.e. let $\alpha^\frown g_i$ be accessible at the next substage.

- Subcase α2.3. $\mathcal{S}_{\alpha,i}$ is inside a gap. Let $t < s$ be the stage at which the gap was open. Close this gap as follows.

◇ Subcase $\alpha 2.3(a)$ (Successful close)

If $C_s \upharpoonright (\theta_\alpha(E_\alpha, C; i)[t]+1) \neq C_t \upharpoonright (\theta_\alpha(E_\alpha, C; i)[t]+1)$, then redefine $\Theta_\alpha^{E_\alpha,C}(i) = 1$ with use $\theta_\alpha(E_\alpha, C; i)[s]$ big, and declare that the gap is closed successfully and that $\mathcal{S}_{\alpha,i}$ is satisfied. Define $\sigma_s = \alpha^\frown g_{i+1}$, and initialize all the strategies with priority lower than σ_s. Go to the next stage.

◇ Subcase $\alpha 2.3(b)$ (Unsuccessful close)

If $C_s \upharpoonright (\theta_\alpha(E_\alpha, C; i)[t] + 1) = C_t \upharpoonright (\theta_\alpha(E_\alpha, C; i)[t] + 1)$, then define $r(\alpha^\frown g_i) = s'$ and enumerate $\theta_\alpha(E_\alpha, C; i)[t]$ into E_α (*this enumeration undefines* $\Theta_\alpha^{E_\alpha,C}(j)$ *for all* $j \geq i$). Request that $\theta_\alpha(E_\alpha, C; i)$ be defined as big. Here s' is the stage at which the gap is open. We say that the gap is closed unsuccessfully. Let $\sigma_{s,t+1} = \alpha^\frown g_i$, i.e. let $\alpha^\frown g_i$ be accessible at the next substage.

Stage $s = 2e+1$ ($e \geq 0$). Let c_s be the number enumerated into C at stage s. Perform the cogap permission (if there exists) as follows: find a substrategy (if any) with the highest priority, α say, an $\mathcal{S}_{\alpha,i}$-strategy, inside a cogap with $c_s \in dom(f_{\alpha,i}^x)$ for some x, with the x-turn in progress. Let $r(\alpha^\frown g_i) = 0$, and declare that the corresponding computation $\Phi_\alpha^B(x)$ is preserved by $\mathcal{S}_{\alpha,i}$. Stop the x-turn and start the $(x+1)$-turn. Define $\sigma_s = \xi$, where ξ is a strategy below $\alpha^\frown g_i$ with lowest priority which has been accessible earlier, and has never been initialized since then. So all the strategies to the right of σ_s are initialized at this stage. If there is no such strategy, do nothing and let $\sigma_s = \sigma_{s-1}$.

This completes the full construction.

2.4 Verification

Let $TP = \liminf_s \sigma_s$, the true path of the construction. In the following, we assume that C is incomputable and that C has a cappable degree. The following lemmas ensure that the construction given above satisfies all the requirements.

Lemma 2.1. *Fix σ on TP. Then:*

1. *σ can be initialized at most finitely often.*

2. *σ has an outcome $\mathcal{O}$ such that $\sigma^\frown\mathcal{O}$ is also on TP.*

3. *If $\sigma^\frown\mathcal{O}$ is on TP, and σ is a $\mathcal{P}$-strategy, then the actions of substrategies of σ can initialize those strategies below $\sigma^\frown\mathcal{O}$ at most finitely often.*

Proof. We prove the claim by induction. Let $\sigma \in TP$ and let σ^- be the immediate predecessor of σ. By the induction hypothesis, there is some stage s_0 after which σ^- cannot be initialized again. There are two cases.

Case 1. *$\sigma^- = \beta$ is an $\mathcal{N}$-strategy.*

If $\sigma = \beta^\frown i$ is on TP, then σ can never be initialized after stage s_0. If $\sigma = \beta^\frown f$ is on TP, then there are only finitely many β-expansionary stages, and hence, there is a stage, $s_1 > s_0$ say, large enough after which no nodes to the left of $\beta^\frown f$ can be accessible again during the even stages. Note that the associated functions f are only extended at even stages, we know that after stage s_1, no nodes to the left of $\beta^\frown f$ can extend the definition of these functions. As a consequence, σ can be initialized at most finitely many more times after s_1 (by cogap permission actions). Note that when σ^- enumerates γ-uses into B, or creates and cancels a link, we do not initialize the strategies below it as these enumerations will not affect the strategies with lower priority (they are using believable computations). Therefore, (1) is true for σ.

In this case, σ is a $\mathcal{P}$-strategy and σ has infinitely many possible outcomes, $g_i, i \in \omega$, and f. If there are only finitely many expansionary stages, then f is the true outcome of σ and $\sigma^\frown f$ is on the true path. Hence (2) is true. Obviously (3) is also true. So we assume that there are infinitely many σ-expansionary stages. We need to argue that there is an outcome g_i such that $\sigma^\frown g_i$ is on TP. We prove it by contradiction.

Suppose not. Then for each $i \in \omega$, $\sigma^\frown g_i$ can be accessible at most finitely often in the construction. Let t_i be the last stage at which $\sigma^\frown g_i$ is accessible. Then we will have that for each $i \in \omega$,

$$\Theta_\sigma^{E_\sigma, C}(i) \downarrow = \Theta_\sigma^{E_\sigma, C}(i)[t_i] = E'_\sigma(i)[t_i] = E'_\sigma,$$

with $\theta_\sigma(i) = \theta_\sigma(i)[t_i]$. Thus we have $E'_\sigma = \Theta_\alpha^{E_\alpha, C}$ and $E'_\sigma \leq_T E_\sigma \oplus C \leq_T \emptyset'$, E_σ is low and C is low-cuppable. This means that C has a noncappable degree, by a characterization of cappable degrees of Ambos-Spies, Jockusch, Shore and Soare in [1]. A contradiction.

So there is a least i such that $\sigma^\frown g_i$ can be visited infinitely often. That is, $\sigma^\frown g_i$ is on TP and hence (2) is true.

(3) is also true for σ as $\sigma^\frown g_i$ itself does not initialize lower strategies at all. So if a strategy below $\sigma^\frown g_i$ is initialized, it must be initialized by a strategy to the left of $\sigma^\frown g_i$. By our assumption on g_i, such initializations can happen at most finitely often.

Case 2. *$\sigma^- = \alpha$ is a $\mathcal{P}$-strategy.*

If $\sigma = \sigma^{-\frown} g_i$ for $i \in \omega$, then there are only i many outcomes of α to the left of σ. By the induction hypothesis, there is a stage, $s_1 > s_0$ say, large enough after which all the strategies $\alpha^\frown g_i$, $j < i$, cannot be accessible at even stages anymore. Therefore, after stage s_1, σ can be initialized only by cogap permission actions. As all the strategies involved in such actions are to the left of σ, there are only finitely many such actions which can initialize σ after stage s_1. Let $s_2 > s_1$ be the last stage at which such a cogap permission action occurs (if any). Then after stage s_2, σ cannot be initialized anymore. If $\sigma = \sigma^{-\frown} f$, then exactly as we have discussed in case 1, there are only finitely many α-expansionary stages, and hence after a particular stage, each α-stage will be a σ-stage (so σ cannot be initialized at even stages), and again, we can argue that after a stage large enough, no cogap permission action can happen to the left of σ, which means that σ cannot be initialized at odd stages. In any case, (1) is true.

As σ in this case is an $\mathcal{N}$-strategy, if there are infinitely many σ-expansionary stages in the construction, then $\sigma^\frown i$ is on TP, and if there are only finitely many σ-expansionary stages, then $\sigma^\frown f$ is on TP. Thus (2) is true for σ.

Note that the $\mathcal{N}$-strategies never initialize other strategies in the construction. □

The following is obvious now.

Lemma 2.2. *TP is infinite.*

We prove in the next two lemmas that each requirement is satisfied along TP.

Lemma 2.3. *For any $e \in \omega$, let β be the $\mathcal{N}_e$-strategy on TP. Suppose that $\Phi_e^{B,W_e} = K \oplus F$, then $\Delta_\beta^{W_e}$ (defined by β) computes K correctly. That is, $\mathcal{N}_e$ is satisfied at β.*

Proof. By Lemma 2.1, there is a stage s_β after which β can no longer be initialized. We assume that $\Phi_e^{B,W_e} \downarrow = K \oplus F$. Then there are infinitely many σ-expansionary stages, and σ has outcome i on TP. First, we have the following claim:

Claim: For $x \in \omega$, $\Delta_\beta^{W_e}(x)$ can be undefined (redefined) at most finitely often.

The claim is obviously true, as we are assuming that $\Phi_e^{B,W_e} \downarrow = K \oplus F$, and for a fixed x, $\delta_\beta(x)$ is changed to a bigger number (of course $\Delta_\beta^{W_e}(x)$ is undefined) only when W_e has changes below the corresponding $\varphi_e(x)$.

So by our construction of Δ_β, $\Delta_\beta^{W_e}$ is well-defined. Now we show that $\Delta_\beta^{W_e}$ computes K correctly.

Let s_x be the last stage at which $\delta_\beta(x)$ is defined. Then $\Delta_\beta^{W_e}(x)[s_x] = K_{s_x}(x)$ and also $\Phi_e^{B,W_e}(2x)[s_x] \downarrow = (K \oplus F)(2x)[s_x]$.

Without loss of generality, suppose that $K(x)$ changes after stage s_x. Then by our assumption on s_x, there is no W_e-changes on $\varphi_e(x)[s_x]$ (otherwise, $\delta_\beta(x)$ would be increased because of such W_e-changes). We now show that B also has no change below $\varphi_e(x)[s_x]$. Suppose not and suppose that a number $z < \varphi_e(x)[s_x]$ is enumerated into B after stage s_x. Then z is a γ-use chosen before stage s_x. As in the construction, when z is selected, several auxiliary numbers, y say, are also selected, and when a $\mathcal{P}$-strategy wants to put z into B, it first puts these auxiliary numbers into F (one by one via links) first, either to make a disagreement between $\Phi^{B,W}$ and $K \oplus F$, or to force a change of the associated W to undefine those $\Delta^W(-)$ defined after the stage at which z is selected. As s_x is a β-expansionary stage, $l(\beta, s_x)$ is bigger than $2y+1$. In particular, $\Phi^{B,W}(2y+1)[s_x] = (K \oplus F)(2y+1)[s_x]$, and $\delta_\beta(x)$ is defined as s_x, which is bigger than $\varphi_e(2y+1)[s_x]$. When y is put into F, a link is created between β and the corresponding $\mathcal{P}$-strategy. If this link is never cancelled later, then $\Phi^{B,W}(2y+1) \neq (K \oplus F)(2y+1)$, which cannot be true, as otherwise σ could not be on TP. So this link is cancelled later, which means that W_e has a change below $\varphi_e(2y+1)[s_x]$ and hence below $\delta_\beta(x)$. This W_e-change undefines $\Delta_\beta^{W_e}(x)$, and when we define it later, we will select the new use bigger. This cannot be true, by the choice of s_x.

Therefore Δ_β is well-defined and computes W_e correctly. Thus $\mathcal{N}$ is satisfied at β. □

Lemma 2.4. *For any $e \in \omega$, let α be the $\mathcal{P}_e$-strategy on TP. Then α satisfies $\mathcal{P}_e$. In particular, for any x, $\Gamma^{B,C}(e,x)$ is defined, and $\lim_x \Gamma^{B,C}(e,x)$ exists, and satisfying*

(a) if $\alpha^\frown f$ is on TP, then $r(\alpha^\frown f) = 0$, and $\Gamma^{B,C}(e,x)$ is defined as 0 for almost all $x \in \omega$. Hence, $Tot^B(e) = 0 = \lim_x \Gamma^{B,C}(e,x)$.

(b) if $\alpha^\frown g_i$ is on TP for some $i \in \omega$, then $\liminf_s r(\alpha^\frown g_i)[s] = 0$ and $\mathcal{S}_{\alpha,i}$ defines $\Gamma^{B,C}(e,x) = 1$ for almost all $x \in \omega$, and ensures that for each $x \in \omega$, the x-turn ends successfully. The latter ensures the totality of Φ_e^B, and hence shows that $Tot^B(e) = 1 = \lim_x \Gamma^{B,C}(e,x)$.

Proof. Suppose first that $\alpha^\frown f$ is on TP. By Lemma 2.1, there is a (least) stage $s_{\alpha,f}$ large enough after which $\alpha^\frown f$ cannot be initialized. Under this outcome, no restraint is imposed during the construction, $r(\alpha^\frown f) = 0$ and hence, by induction, $R(\alpha^\frown f)$ is finite. Let x_0 be the least number such that $\Gamma^{B,C}(e,x)$ has no definition at stage $s_{\alpha,f}$, or $\Gamma^{B,C}(e,x)$ is currently defined by a strategy to the right of α with $\gamma(e,x) > R(\alpha^\frown f)$. Then, after stage $s_{\alpha,f}$, any α-stage is also an $\alpha^\frown f$-stage, and at these stages, $\Gamma^{B,C}(e,x)$ is defined for all $x \geq x_0$ with $\Gamma^{B,C}(e,x) \downarrow= 0$ to make $\lim_x \Gamma^{B,C}(e,x) = 0 = Tot^B(e)$, satisfying $\mathcal{P}_e$.

The basic idea of this case is quite simple, as Φ_e^B is not total (only finitely many expansionary stages) and that $\alpha^\frown f$ always defines $\Gamma^{B,C}(e,x) = 0$ for $x \geq x_0$. Of course, when we want to undefine $\Gamma^{B,C}(e,x)$, which is defined by a strategy to the right of $\alpha^\frown f$, links to higher priority $\mathcal{N}$-strategies are created and cancelled first, which in some sense will delay, but not affect, the enumeration of the current value of $\gamma(e,x)$ into B.

In this case, once $\alpha^\frown f$ (re)defines $\Gamma^{B,C}(e,x)$, $\gamma(e,x)$ can be lifted up to a new number only when a strategy to the left of $\alpha^\frown f$ wants to undefine it. Note that a C-change below $\gamma(e,x)$ also undefine $\Gamma^{B,C}(e,x)$, but as indicated in the construction, when $\Gamma^{B,C}(e,x)$ is defined again, $\gamma(e,x)$ is defined the same as before, and hence such a process can happen at most finitely many times. By our choice of $s_{\alpha,f}$, we know that $\gamma(e,x)$ cannot be changed later, and hence after a stage large enough, $\Gamma^{B,C}(e,x)$ cannot be undefined, which means that it is defined.

Now we assume that $\alpha^\frown g_i$ is on TP for some $i \in \omega$. In this case, α will open and close gaps infinitely often. As $r(\alpha^\frown g_i)$ is set to be 0 when we open a gap, $\liminf_s r(\alpha^\frown g_i)[s] = 0$ and hence, again, by induction, $R(\alpha^\frown g_i)$ is finite. Similar to the case above, with $s_{\alpha,f}$ replaced by $s_{\alpha,i}$, we can prove that $\mathcal{S}_{\alpha,i}$ will define $\Gamma^{B,C}(e,x)\downarrow= 1$ for all $x \geq x_0$. Therefore, $\lim_x \Gamma^{B,C}(e,x) = 1$.

We now show that for each $x \in \omega$, the x-turn of $\mathcal{S}_{\alpha,i}$ ends at a cogap via a C permission. Suppose not. Let x be the least number such that the x-turn never stops. By the assumption that $\alpha^\frown g_i$ is on TP, the $\mathcal{S}_{\alpha,i}$-substrategy opens and closes infinitely many gaps in the x-turn. Of course, each gap is closed unsuccessfully, and C does not change below $f^x_{\alpha,i}$ during any cogap. C does not change below $f^x_{\alpha,i}$ during any gap either, as otherwise, such a C-change will make the corresponding gap be closed successfully. Since we always undefine $\Theta^{E_\alpha,C}_\alpha(i)$ by enumerating the corresponding $\theta_\alpha(E_\alpha,C)$-uses into E_α when a gap is closed unsuccessfully, $\theta_\alpha(E_\alpha,C;i)$ is defined infinitely often in increasing order. Thus $f^x_{\alpha,i}$ is extended to a total function. From this, it would follow that C is computable, which is a contradiction.

As when the x-turn ends, a C-change appears and we can lift the γ-uses to bigger numbers, so that the computation $\Phi^B_e(x)$ cannot be injured by the definition of Γ. However, even after this, $\Phi^B_e(x)$ can still be injured by strategies below $\alpha^\frown g_i$ later. We need to show that such such injuries can happen at most finitely often.

If the x-turn stops at stage s_0, then at stage s_0, $\mathcal{S}_{\alpha,i}$ implements the cogap permission and $\gamma(e,x)$ is increased to a number bigger than $\varphi_e(x)$. Suppose that the x-turn is injured at stage s_1, then $\mathcal{S}_{\alpha,i}$ will continue to run the x-turn trying to preserve $\Phi^B_e(x)$. Fortunately, such injuries can happen at most finitely often, because only finitely many strategies below $\alpha^\frown g_i$, and only finitely many numbers, z say, of these strategies can have α-bound less than x, and only the enumeration of these γ-uses $\gamma(-,z)$ into B can injure the x-turn. This shows that $\Phi^B_e(x)\downarrow$.

By induction on x, we have that Φ^B_e is total, and hence, $Tot^B(e) = 1 = \lim_x(e,x)$.

$\mathcal{P}_e$ is satisfied. $\square$

This completes the proof of Theorem 1. $\diamond\diamond$

Chapter 3

INFIMA AND DIAMOND EMBEDDINGS

In the Ershov hierarchy, the d.c.e. sets are the closest class to the c.e. sets. The study of the interactions between $\mathbf{R}$ and $\mathbf{D}_2$ has generic significances to the exploration of Ershov hierarchy, since they are the first two levels in Ershov hierarchy and most structural differences between $\mathbf{R}$ and $\mathbf{D}_2$ can be generalized to higher levels in Ershov hierarchy. For example, the fact that, each nonzero d.c.e. degree bounds a nonzero c.e. degree, can be generalized to that, each nonzero n-c.e. degree bounds a nonzero c.e. degree. In the following three chapters, we will investigate the structural differences between $\mathbf{R}$ and $\mathbf{D}_2$ from three aspects: infima, diamond lattice embedding and cupping property.

3.1 Introduction

In this chapter, we consider the infima of d.c.e. degrees in the different levels of Ershov hierarchy, and provide an interesting diamond lattice embedding in $\mathbf{D}_2$. In [44], Kaddah showed that there are two d.c.e. degrees $\mathbf{a}, \mathbf{b}$ forming a minimal pair in the d.c.e. degrees, but not in the Δ^0_2 degrees. Kaddah's result shows that Lachlan's theorem, stating that the infimum of two c.e. degrees in the c.e. degrees coincides with that in the Δ^0_2 degrees, cannot be generalized to the d.c.e. degrees. This result can be deduced from following theorem.

Theorem 3 (Liu and Wu [55]) *There are c.e. degrees* $\mathbf{a}, \mathbf{b}$*, d.c.e. degrees* $\mathbf{c}, \mathbf{d}$ *and a nonzero* ω*-c.e. degree* $\mathbf{e} \leq \mathbf{c}, \mathbf{d}$ *such that* $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$*,* $\mathbf{a}, \mathbf{b}$ *form a minimal pair, and that* $(\mathbf{a}, \mathbf{c})$*,* $(\mathbf{b}, \mathbf{d})$ *are isolation pairs. Consequently,* $\{\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ *forms a diamond in the d.c.e. degrees, but not in the* Δ^0_2 *degrees.*

In consequence, the diamond embedding $\{\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ is different from the one first constructed by Downey in [25] where two atoms form a minimal pair in the Δ^0_2 degrees. To obtain this, we will construct c.e. degrees $\mathbf{a}, \mathbf{b}$, d.c.e. degrees $\mathbf{c} > \mathbf{a}, \mathbf{d} > \mathbf{b}$ and a nonzero ω-c.e. degree $\mathbf{e} \leq \mathbf{c}, \mathbf{d}$ such that (1) $\mathbf{a}, \mathbf{b}$ form a minimal pair, (2) $\mathbf{a}$ isolates $\mathbf{c}$, and (3) $\mathbf{b}$ isolates $\mathbf{d}$. From this, we can have that $\mathbf{c}, \mathbf{d}$ form a minimal pair in the d.c.e. degrees, and Kaddah's result follows immediately. In our construction, we apply Kaddah's original idea to make $\mathbf{e}$ below both $\mathbf{c}$ and $\mathbf{d}$. Our construction allows us to separate the minimal pair argument ($\mathbf{a} \cap \mathbf{b} = \mathbf{0}$), the splitting of $\mathbf{0}'$ ($\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$), and the non-minimal pair of $\mathbf{c}, \mathbf{d}$ (in the Δ^0_2 degrees), into several parts, to avoid direct conflicts that could be involved if only $\mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ are constructed.

In addition, our construction of $\mathbf{a}, \mathbf{b}$ in Theorem 3 can be combined with the standard highness strategy to make $\mathbf{a}, \mathbf{b}$ (and hence $\mathbf{c}, \mathbf{d}$) high. For the construction of high minimal pairs, [80] is a good reference.

3.2 Requirements and basic strategies

To prove Theorem 3, we construct c.e. sets A, B, d.c.e. sets C, D, an ω-c.e. set E and partial functionals $\Gamma, \Omega_1, \Omega_2$ satisfying the following requirements:

$\mathcal{G} : K = \Gamma^{C,D}$;

$\mathcal{S} : E = \Omega_1^C = \Omega_2^D$;

$\mathcal{M}_e : \Phi_e^{A,C} = W_e \rightarrow \exists \Delta_e (\Delta_e^A = W_e)$;

$\mathcal{N}_e : \Phi_e^{B,D} = W_e \rightarrow \exists \Theta_e (\Theta_e^B = W_e)$;

$\mathcal{P}_e : C \neq \Phi_e^A$;

$\mathcal{Q}_e : D \neq \Phi_e^B$;

$\mathcal{R}_e : \Phi_e^A = \Phi_e^B = g$ total $\to g$ computable;

$\mathcal{T}_e : E \neq \Phi_e$,

where $e \in \omega$, $\{(\Phi_e, W_e) : e \in \omega\}$ is an effective enumeration of all pairs (Φ, W) such that Φ is a partial computable functional and W is a computably enumerable set. K is a fixed creative set.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ be the Turing degrees of $A, B, A \oplus C, B \oplus D, E$ respectively. The $\mathcal{M}$ and $\mathcal{P}$-requirements guarantee that $(\mathbf{a}, \mathbf{c})$ is an isolation pair. The $\mathcal{N}$ and $\mathcal{Q}$-requirements guarantee that $(\mathbf{b}, \mathbf{d})$ is an isolation pair. Thus, $\mathbf{c}$ and $\mathbf{d}$ are both proper d.c.e. degrees and $\mathbf{a}$ and $\mathbf{b}$ are nonzeros. By $\mathcal{G}$, we have $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$, which ensures that $\mathbf{c}$ and $\mathbf{d}$ are incomparable.

The existence of E satisfying $\mathcal{S}$ and all $\mathcal{T}$-requirements ensures that $\mathbf{c}$ and $\mathbf{d}$ do not form a minimal pair in the Δ_2^0 degrees.

The $\mathcal{R}$-requirements guarantee that $\mathbf{a}, \mathbf{b}$ are incomparable. This implies that $\mathbf{a}, \mathbf{b}$ are both nonzero and that $\mathbf{c}$ and $\mathbf{d}$ also form a minimal pair in the d.c.e. degrees. To see this, suppose that $\mathbf{f}$ is a d.c.e. degree below $\mathbf{c}$ and $\mathbf{d}$. We want to show that $\mathbf{f}$ is $\mathbf{0}$. Suppose not. Then there is a nonzero c.e. degree below $\mathbf{f}$, $\mathbf{x}$ say, and hence below both $\mathbf{c}$ and $\mathbf{d}$. Since $\mathbf{a}$ isolates $\mathbf{c}$ and $\mathbf{b}$ isolates $\mathbf{d}$, $\mathbf{x}$ is also below both $\mathbf{a}$ and $\mathbf{b}$. As $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair, $\mathbf{x} = \mathbf{0}$. A contradiction.

Our construction will satisfies all the requirements, and therefore, the corresponding degrees are those wanted in Theorem 3.

3.2.1 The $\mathcal{G}$-strategy

The $\mathcal{G}$-strategy is to code K into $C \oplus D$, which proceeds as follows:

1. If there is an x such that $\Gamma^{C,D}(x) \downarrow \neq K(x)$, then let k be the least such x, enumerate $\gamma(k)$ into D, and for any $x \geq k$, let $\Gamma^{C,D}(x)$ be undefined.

2. If k is the least number x such that $\Gamma^{C,D}(x)\uparrow$, then define $\Gamma^{C,D}(k) = K(k)$ with $\gamma(k)$ fresh.

Note that the $\mathcal{G}$-strategy only enumerates numbers into D. In the construction, γ-uses will be enumerated into C infinitely many often, which will guarantee that D is incomplete.

The use function γ of Γ will have the following properties:

1. Whenever we define $\gamma(k)$, we define it as a big number;

2. For any k, s, if $\Gamma^{C,D}(k)[s]\downarrow$, then $\gamma(k)[s] \notin C_s \cup D_s$;

3. For any x, y, if $x < y$, and $\gamma(y)\downarrow$, then $\gamma(x)\downarrow$, and $\gamma(x) < \gamma(y)$;

4. If $\Gamma^{C,D}(k)[s]\downarrow= 0$, and $k \in K_{s+1} - K_s$, then there is an $n \leq k$ such that $\gamma(n)[s] \in C - C_s$ or $\gamma(n)[s] \in D - D_s$;

5. $\Gamma^{C,D}(x)$ is undefined at stage s iff at stage s, there is an $y \leq x$ such that $\gamma(y)$ is enumerated into C or D.

If $(1) - (5)$ are met and $\Gamma^{C,D}$ is total, then $\Gamma^{C,D} = K$. $\mathcal{G}$ is satisfied.

3.2.2 The $\mathcal{P}$ and $\mathcal{Q}$-strategies

These strategies are just the well-known Friedberg-Muchnik diagonalization. We present the main idea of the $\mathcal{P}$-strategies for illustration.

Let α be a $\mathcal{P}$-strategy. α works as follows:

1. Appoint an attacker, x say as a big number.

2. Wait for a stage s at which $\Phi_e^A(x)[s]\downarrow= 0$.

3. Enumerate x into C and stop.

α has two possible outcomes, $0, 1$ with $0 <_L 1$, where 0 indicates that α arrives at step 3 eventually, and 1 indicates that α keeps waiting at step 2.

3.2.3 The $\mathcal{S}$ and $\mathcal{T}$-strategies

The $\mathcal{S}$-strategy and $\mathcal{T}$-strategies are designed to satisfy the $\mathcal{S}$ and $\mathcal{T}$-requirements respectively, which will ensure that E is computable in both C and D, and that E is incomputable. Note that $\mathcal{S}$-requirement is a global requirement.

To satisfy the $\mathcal{S}$-requirement, we construct two p.c. functionals Ω_1, Ω_2 such that $E = \Omega_1^C = \Omega_2^D$.

Let β be a $\mathcal{T}$-strategy. The basic action of β is fairly easy. That is, choose x as an attacker and wait for $\Phi_e(x)$ to converge to 0. If $\Phi_e(x)$ does not converge to 0, then β is satisfied. Otherwise, that is, $\Phi_e(x)$ converges to 0, then we put x into E and the corresponding Ω-uses, $\omega_1(x)$, $\omega_2(x)$, into C and D respectively. β is again satisfied, and $\Omega_1^C(x)$ and $\Omega_2^D(x)$ are both undefined and can be defined as 1 later. Thus, β's action is consistent with $\mathcal{S}$.

Note that when we put x into E, $\omega_1(x)$ and $\omega_2(x)$ are put into C and D respectively. As we will see soon, the $\mathcal{M}$ and $\mathcal{N}$-strategies are quite sensitive to the enumerations into C and D, and can act by removing numbers from C or D. Suppose that $\omega_1(x)$ is removed from C, by an $\mathcal{M}$-strategy with priority higher than β, then $\Omega_1^C(x)$ becomes incorrect because this removing makes $\Omega_1^C(x) = 0$. . If it happens, to ensure that $\mathcal{S}$ is satisfied, we have to take x out of E. Now as $\Omega_2^D(x)$ is currently defined as 1, we need to undefine it by putting $\omega_2(x) - 1$ into D.

Also note that when we take numbers out of C, we also put some small number into A, which ensures that the action of this $\mathcal{M}$-strategy is consistent with those $\mathcal{M}$-strategies with higher priority. However, the enumeration of this $\omega_2(x) - 1$ can lead some $\mathcal{N}$-strategy with higher priority to take the same action, which will take this $\omega_2(x) - 1$ out of D. Consequently, we need to put x back to E. Now we need to put a number $\omega_1(x) - 2$ into C, to undefine $\Omega_1^C(x)$.

This process can be iterated several times. Fortunately, as we will see soon, whenever it happens one more time, an $\mathcal{M}$ or $\mathcal{N}$-strategy with priority higher than β (along β), is satisfied, and hence this process can be iterated at most l many times, where l is the length of β.

With this in mind, when we define $\Omega_1^C(x)$ and $\Omega_2^D(x)$ for the first time, we not only define the uses $\omega_1(x)$ and $\omega_2(x)$ as big numbers, we also let $\omega_1(x) - l$ and $\omega_2(x) - l$ be big. The interval $[\omega_2(x)-l, \omega_2(x)]$ is reserved for the future definition and rectification of $\Omega_1^C(x)$ and $\Omega_2^D(x)$. As explained above, $[\omega_2(x)-l, \omega_2(x)]$ is large enough, and the $\mathcal{S}$-strategy, the $\mathcal{T}$-strategies are consistent with the $\mathcal{M}$ and $\mathcal{N}$-strategies.

Remark: In the definition of Ω_1^C and Ω_2^D, once $\Omega_1^C(x)$ and $\Omega_2^D(x)$ are defined, the corresponding uses $\omega_1(x)$ and $\omega_2(x)$ will keep unchanged during the whole construction. $\Omega_1^C(x)$ or $\Omega_2^D(x)$ is undefined only when a number in the interval $[\omega_2(x)-l, \omega_2(x)]$ is enumerated into C or D respectively.

3.2.4 The $\mathcal{R}$-strategies

An $\mathcal{R}$-strategy, τ say, attempting to satisfy an $\mathcal{R}$-requirement, $\mathcal{R}_e$ say, is a standard minimal pair argument. That is, τ tries to preserve the computations up to the largest agreement between Φ_e^A and Φ_e^B until a longer agreement appears.

Define the length function of agreement between Φ_e^A and Φ_e^B as follows:

$$\ell_{\mathcal{R}}(\tau, s) = \max\{x : \forall y < x(\Phi_e^A(y)[s] \downarrow = \Phi_e^C(y)[s] \downarrow)\},$$
$$m_{\mathcal{R}}(\tau, s) = \max\{0, \ell_{\mathcal{R}}(\tau, t) : t < s \;\&\; t \text{ is an } \tau-\text{stage}\}.$$

Say that stage s is τ-expansionary if $s = 0$ or $\ell_{\mathcal{R}}(\tau, s) > m_{\mathcal{R}}(\tau, s)$.

τ has two outcomes, $0, 1$, with $0 <_{\mathrm{L}} 1$, where 0 indicates that there are infinitely many τ-expansionary stages and 1 for the other case.

Suppose that $\Phi_e^A = \Phi_e^B = g$ is total, then there are infinitely many τ-expansionary stages. We compute g as follows.

Given x. Find the least τ-expansionary stage, s_1 say, such that s_1 with $\ell_{\mathcal{R}}(\tau, s)[s_1] > x$. We claim that $g(x) = \Phi_e^A(x)[s_1] = \Phi_e^C(x)[s_1]$.

Let $s_1 < s_2 < \cdots < s_i < s_{i+1} < \cdots$ be the sequence of all τ-expansionary stages. Since at stage $s_1 + 1$, numbers can only be enumerated into at most one of A and B, we have:

$$A_{s_1} \upharpoonright s_1 = A_{s_1+1} \upharpoonright s_1 \;\text{ or }\; V_{s_1} \upharpoonright s_1 = B_{s_1+1} \upharpoonright s_1.$$

Note that any number enumerated into $A \cup B$ between stages $s_1 + 1$ and s_2 are bigger than s_1 (*because they are associated with strategies on the right of* $\tau^\frown 0$ *when we work* *on a tree, and these strategies were initialized at stage* $s_1 + 1$), we have

$$A_{s_2} \upharpoonright s_1 = A_{s_1} \upharpoonright s_1 \text{ or } B_{s_2} \upharpoonright s_1 = B_{s_1} \upharpoonright s_1.$$

Therefore, $g(x)[s_2] = g(x)[s_1]$.

By induction, we can prove that for any $i \geq 1$,

$$A_{s_i} \upharpoonright s_i = A_{s_{i+1}} \upharpoonright s_i \text{ or } B_{s_i} \upharpoonright s_i = B_{s_{i+1}} \upharpoonright s_i,$$

and $g(x)[s_{i+1}] = g(x)[s_i]$. Thus, for any $i \geq 1$, $g(x)[s_i] = g(x)[s_1]$, we hence $g(x) = g(x)[s_1]$. g is computable.

3.2.5 The $\mathcal{M}$ and $\mathcal{N}$-strategies

The $\mathcal{M}$-strategies and $\mathcal{N}$-strategies are now known as the isolation strategy. We use the $\mathcal{M}$-strategies to illustrate the main idea. The $\mathcal{N}$-strategies are just the same.

An $\mathcal{M}$-strategy, η, is to satisfy an $\mathcal{M}$-requirement, $\mathcal{M}_e$ say. Define the length function of the agreement between $\Phi_e^{A,C}$ and W_e as follows:

$$\ell_{\mathcal{M}}(\eta, s) = \max\{x < s : (\forall y < x)(\Phi_e^{A,C}(y)[s] \downarrow= W_{e,s}(y))\},$$

$$m_{\mathcal{M}}(\eta, s) = \max\{0, \ell_{\mathcal{M}}(\eta, t) : t < s \text{ and } t \text{ is an } \eta\text{-stage}\}.$$

Say that a stage s is η-expansionary, if $s = 0$ or $\ell_{\mathcal{M}}(\eta, s) > m_{\mathcal{M}}(\eta, s)$.

Generally, at an η-expansionary stage s, we define $\Delta_\eta^A(y)[s] = W_{e,s}(y)$ with $\delta_\eta(y)[s+1] = s$ for $y < \ell_{\mathcal{M}}(\eta, s)$, if $\Delta_\eta^A(y)[s]$ is not defined.

In the construction, the numbers enumerated into C, z say, by a $\mathcal{T}$-strategy, or an $\mathcal{N}$-strategy, (if β is an $\mathcal{N}$-strategy, then z can be enumerated into D by the $\mathcal{G}$-strategy,) may injure the current computation of $\Phi_e^{A,C}(x)[s]$ and consequently, lift the use $\varphi_e(A, C; x)$ to a bigger number. This provides chances for $W_e(x)$ to change, making $\Delta_\eta^A(x)$ incorrect. Suppose that $W_e(x)$ changes at stage s', then removing all these numbers, z, out of C will recover the computation $\Phi_e^{A,C}(x)$ to $\Phi_e^{A,C}(x)[s]$. Such an action creates an inequality between $\Phi_e^{A,C}$ and W_e at x. W_e is c.e., x remains in

W_e. If we preserve the computation $\Phi_e^{A,C}(x)$ from now on, then we have the following inequality,

$$\Phi_e^{A,C}(x) = \Phi_e^{A,C}(x)[s] = W_{e,s}(x) = 0 \neq 1 = W_e(x),$$

and $\mathcal{M}_e$ is satisfied. This method is referred as the isolation argument.

If $\Phi_e^{A,C}$ is total and $\Phi_e^{A,C} = W_e$, then η has no chance to initiate the isolation argument. Thus, if we can ensure that Δ_η^A is total, then Δ_η^A computes W_e correctly.

We are now ready to consider the consistency between η and other strategies.

First, consider the consistency between η and another $\mathcal{M}$-strategy, η' say. When numbers, z say, are extracted from C by η, some computations involved in the η'-strategy are destroyed, which may lead to the incorrectness of $\Delta_{\eta'}^A$. Such incorrectnesses cannot be corrected by enumerating z into C again, because we are making C d.c.e.. To avoid this, whenever η executes the isolation argument at y (extracting z out of C), we put s_y, the stage at which $\Delta_\eta^A(y)$ was defined, into A to undefine $\delta_{\eta'}(y')$ which was defined from stage s_y on. Clearly, such an action does no harm to those $\delta_\eta(y'')$ defined before stage s_y, and hence η is consistent with η'.

Now we consider the consistency between η and the $\mathcal{S}$ and $\mathcal{G}$-strategies.

When we remove numbers from C at stage s', we know that that these numbers, z above, can be a γ-marker, or an ω_1-marker. Simply taking z out may lead to the failure of the $\mathcal{G}$-strategy or the $\mathcal{S}$-strategy. On the other hand, since $\mathcal{G}$ and $\mathcal{S}$ has the highest priority, the enumeration of γ-markers or ω_1-markers into C may destroy the computation $\Phi_e^{A,C}(x)$ and hence injure the isolation argument described above. How can we get around of these problems? The idea is that when we remove numbers from C, we put numbers into D to make sure that Γ and Ω_1, Ω_2 are all well-defined.

If z above is an ω_1-marker, $\omega_1(x)$ say, then as described in the $\mathcal{T}$-strategies, when we remove z out of C, we enumerate $z-1$, into D. Consequently, if x is currently in E, then we will take x out of E, and if x is currently not in E, then we put it into E.

We now show that $\Omega_1^C(x)$ is well-defined, and equals to $E(x)$. Let β be the $\mathcal{T}$-strategy choosing x as its attacker. Then when $\Omega_1^C(x)$, $\Omega_2^D(x)$ are first defined, at stage s_0 say, $\omega_1(x)[s_0]$ and $\omega_1(x)[s_0]$ to be the same, and an interval $[\omega_1(x)[s_0]-|\beta|, \omega_1(x)[s_0]]$ is reserved for the definition of $\Omega_1^C(x)$, $\Omega_2^D(x)$. When we first put x into E (β acts), at stage s_1 say, $\omega_1(x)[s_0]$, $\omega_2(x)[s_0]$ are put into C and D respectively, to undefine $\Omega_1^C(x)$ and $\Omega_2^D(x)$, and next time, at stage s_2, when we define $\Omega_1^C(x)$ and $\Omega_2^D(x)$ again, at stage s_2, we redefine $\Omega_1^C(x)$, $\Omega_2^D(x)$ with uses $\omega_1(x)[s_2] = \omega_1(x)[s_1]$, $\omega_2(x)[s_2] = \omega_2(x)[s_1]$.

Now if an $\mathcal{M}$-strategy $\beta \subset \alpha$ performs an isolation argument at stage s_3, and $\omega_1(x)[s_0]$ is taken out of C, then $\Omega_1^C(x)$ is recovered to 0, since $C \upharpoonright (\omega_1(x)[s_0] + 1)$ is recovered to $C[s_0] \upharpoonright (\omega_1(x)[s_0] + 1)$. Thus, at this stage, we need to remove x from E.

To ensure that $\Omega_2^D(x)$ computes $E(x)$ correctly, we also need to put $\omega_2(x)[s_0] - 1$ into D to undefine $\Omega_2(x)$. Here, since we need to ensure the consistency between the $\mathcal{S}$, $\mathcal{T}$-strategies and all the $\mathcal{M}$, $\mathcal{N}$-strategies, when we take numbers from one of C and D, we put numbers into the other one.

Note that the minimal pair construction of A and B prevents us from removing numbers from C and D simultaneously, since some (small) numbers need to be put into A or B respectively when numbers are removed from C or D, according to our description above.

The enumeration of numbers into D at stage s_3 may lead to a diagonalization of an $\mathcal{N}$-strategy $\eta' \subset \eta$, at stage s_4. Note that s_3 is a η'-expansionary stage, so at stage s_4, we can do the diagonalization by removing those numbers (including $\omega_2(x)[s_0] - 1$) enumerated into D from stage s_3, and enumerating s_3 into B for the sake of the consistency between η' and those $\mathcal{N}$-strategies with higher priority. Since there are at most $|\beta|$ many such $\mathcal{M}$ and $\mathcal{N}$ strategies above β, which can enumerate or extract the numbers in the interval $[\omega_1(x)[s_0] - |\beta|, \omega_1(x)[s_0]]$, and once such a strategy acts, it will not act again on these numbers, the size of the interval is large enough for the $\mathcal{M}$ and $\mathcal{N}$ strategies' actions, and the $\mathcal{M}$ and $\mathcal{N}$-strategies are consistent with the $\mathcal{S}$-strategy.

Now we consider the consistency between the $\mathcal{M}$ and $\mathcal{N}$-strategies and the $\mathcal{G}$-strategy. The main idea is to introduce a parameter $k(\eta)$ as a threshold for the enumeration of γ-markers. $k(\eta)$ works as follows:

1. Define $k(\eta)$ as a fresh number.

2. If some $n < k(\eta)$ enters K, then reset η by undefining all parameters of η,

except $k(\eta)$.

(Thus, $k(\eta)$ is fixed, and η can be reset in this way only finitely often.)

3. When η executes the isolation argument, then also enumerate $\gamma(k(\eta))$ into D, and remove all the numbers.

4. If η's request is confirmed, then perform the disagreement argument as described above.

The enumeration of $\gamma(k(\eta))[s]$ into D allows us to redefine $\Gamma^{C,D}(y)$ again. Also since all $\gamma(n)$ with $n \geq k(\eta)$ are lifted above the use $\varphi_e(x)[s]$, the disagreement is preserved forever and $\mathcal{M}_e$ is satisfied.

η has three outcomes, $0 <_{\mathrm{L}} d <_{\mathrm{L}} 1$, where d stands for the fact that the isolation argument succeeds, and 0 and 1 for the cases of whether or not there are infinitely many η-expansionary stages.

This completes the description the $\mathcal{M}$ and $\mathcal{N}$ strategies, and the consistency between these strategies and the $\mathcal{G}$ and $\mathcal{S}$ strategies.

We are ready to give the whole construction.

3.3 Construction

The construction proceeds on a priority tree, T say, which is defined effectively as follows.

First define the *priority* of the requirements as follows:

$$\mathcal{G} < \mathcal{S} < \mathcal{M}_0 < \mathcal{N}_0 < \mathcal{P}_0 < \mathcal{Q}_0 < \mathcal{R}_0 < \mathcal{T}_0 < \mathcal{M}_1 < \mathcal{N}_1 < \mathcal{P}_1 < \mathcal{Q}_1 < \mathcal{R}_1$$
$$< \mathcal{T}_1 < \cdots < \mathcal{M}_n < \mathcal{N}_n < \mathcal{P}_n < \mathcal{Q}_n < \mathcal{R}_n < \mathcal{T}_n < \cdots ,$$

where for any requirements, X, Y say, if $X < Y$, then X has priority higher than Y.

For $\xi \in T$, say that requirement X is *satisfied at* ξ, if there is an X-strategy τ such that $\tau \subset \xi$. $\mathcal{G}$ and $\mathcal{S}$ are global requirements, and we do not list them on T.

Definition 1 (1) Define the root node λ as an $\mathcal{M}_0$-strategy;

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy;

(3) For $\xi \in T$, ξ works for the highest priority requirement which is not satisfied at ξ.

In the construction, if ξ is initialized, then any parameters will be cancelled. If ξ is an $\mathcal{M}$ and an $\mathcal{N}$-strategy, then say that ξ is reset, if we cancel all parameters of ξ, except $k(\xi)$. If $\xi < \xi'$ and ξ is reset or initialized, then ξ' will be initialized simultaneously and automatically.

The Construction

Without loss of generality, suppose that K is enumerated at stages $3k+1$, $k \in \omega$, and that exactly one element is enumerated into K at such stage. We will define $\Gamma^{C,D}$ at stages $3k$ and stages $3k+1$, $k \in \omega$, and leave stages $3k+2$, $k \in \omega$, for the definition of Ω_1^C, Ω_2^D. The following is the stage-by-stage construction:

`Stage` 0: Set $A = B = C = D = E = \emptyset$, and initialize all nodes.

`Stage` $s+1$:

(I) $s = 3k$. Let k be the number in $K_{s+1} - K_s$. There are two cases.

1. If $\Gamma^{C,D}(k) \downarrow$, then
   - Enumerate $\gamma(k)[s]$ into D. *(Note that at these stages, the γ-markers are always enumerated into D.)*
   - For any strategy ξ, if $k < k(\xi)$, then reset ξ.
   - Go to the next stage.
2. Otherwise, let x be the least y such that $\Gamma^{C,D}(y) \uparrow$, define $\Gamma^{C,D}(x) = K(x)$ with $\gamma(x)$ fresh, and go to the next stage.

(II) $s = 3k+1$. At these stages, we define Ω_1^C and Ω_2^D. Let x be the least y such that $\Omega_1^C(y)$ or $\Omega_2^D(y)$ are not defined. Suppose that $\Omega_1^C(x)$ is not defined. The case that $\Omega_2^D(x)$ can be defined similarly. Then define $\Omega_1^C(x) = E(x)$ with $\omega_1(x)$ as follows:

1. If $\Omega_1^C(x)$ has been defined before, at stage s_x say, then define $\omega_1(x) = \omega_1(x)[s_x]$, with the same use block.

2. Otherwise, see whether x is appointed as a witness of a $\mathcal{T}$-strategy β. If yes, then define $\Omega_1^C(x) = \Omega_2^D(x) = E(x)$ with use $\omega_1(x) = \omega_2(x)$ such that $\omega_1(x) - |\beta|$ is fresh, and declare that $[\omega_1(x) - |\beta|, \omega_1(x)]$, where $|\beta|$ is the length of β, is an interval reserved for the definition of $\Omega_1^C(x)$ and $\Omega_2^D(x)$. We will refer to the numbers in this interval as ω_1 and ω_2 markers. If no, then just define $\Omega_1^C(x) = \Omega_2^D(x) = E(x)$ with use $\omega_1(x) = \omega_2(x)$ fresh.

Go to the next stage.

(III) $s = 3k + 2$.

Say that a strategy ξ is *visited at* stage $s + 1$, if ξ is eligible to act at a substage t of stage $s + 1$. First, let λ, the root node, be eligible to act at substage 0.

`Substage` t: Let ξ be eligible to act at substage t. If $t = s+1$, then define $\sigma_{s+1} = \xi$, initialize all those $\zeta \not\leq \sigma_{s+1}$, and go to the next stage.

If ξ is an $\mathcal{M}$ or $\mathcal{N}$-strategy and $k(\xi)[s] \uparrow$, then define $k(\xi)[s+1]$ as a fresh number, define $\sigma_{s+1} = \xi$, initialize all $\zeta \not\leq \sigma_{s+1}$, and go to the next stage.

Otherwise, there are four cases:

`Case 1.` $\xi = \alpha$ is a $\mathcal{P}$ or a $\mathcal{Q}$-strategy. We assume that α is a $\mathcal{P}$-strategy. The case when α is a $\mathcal{Q}$-strategy is almost the same, with A and C being changed to B and D respectively.

α1. If $x(\alpha) \uparrow$, then define $x(\alpha)$ as a fresh number, define $\sigma_{s+1} = \alpha$, initialize all $\zeta \not\leq \sigma_{s+1}$, and go to the next stage.

α2. If $x(\alpha) \downarrow$ and $x(\alpha) \in C$, then let $\alpha^\frown 0$ be eligible to act at the next substage;

α3. If $x(\alpha) \downarrow, \Phi_\alpha^A(x(\alpha)) \downarrow = C(x(\alpha)) = 0$, then enumerate $x(\alpha)$ into C, define $\sigma_{s+1} = \alpha$, initialize all $\zeta \not\leq \sigma_{s+1}$, and go to the next stage.

α4. Otherwise, let $\alpha^\frown 1$ be eligible to act at the next substage.

`Case 2.` $\xi = \beta$ is a $\mathcal{T}$-strategy.

β1. If $x(\beta)\uparrow$, then define $x(\beta)$ as a fresh number, define $\sigma_{s+1} = \tau$, initialize all $\zeta \not\leq \sigma_{s+1}$, and go to the next stage.

β2. If $x(\beta)\downarrow$ and $x(\beta) \in E$, then let $\beta^\frown 0$ be eligible to act at the next substage;

β3. If $x(\beta)\downarrow, \Phi_\beta(x(\beta))\downarrow = E(x(\beta)) = 0$, then enumerate $x(\beta)$ into E. If $\omega_1(x(\beta))$ and $\omega_2(x(\beta))$ are defined, then also enumerate $\omega_1(x(\beta))$ and $\omega_2(x(\beta))$ into C and D respectively to undefine $\Omega_1^C(x(\beta))$ and $\Omega_2^D(x(\beta))$. Define $\sigma_{s+1} = \beta$, initialize all $\zeta \not\leq \sigma_{s+1}$, and go to the next stage.

β4. Otherwise, i.e., $x(\beta)\downarrow, \Phi_\beta(x(\beta))[s+1]\uparrow$ or $\Phi_\beta(x(\beta))\downarrow \neq 0$, let $\beta^\frown 1$ be eligible to act at the next substage.

`Case 3.` $\xi = \eta$ is an $\mathcal{M}$ or an $\mathcal{N}$-strategy. We assume that η is an $\mathcal{M}$-strategy, as an $\mathcal{N}$-strategy works in the same way, only with A, C, Δ changed to B, D, Θ respectively.

η1. If $s+1$ is η-expansionary and Δ_η^A is correct, then for any $y < \ell_M(\eta, s)$ with $\Delta_\eta^A(y)[s]\uparrow$, define $\Delta_\eta^A(y)[s+1] = W_{\eta,s+1}(y)$ with use $\delta_\eta(y) = s+1$. Let $\eta^\frown 0$ be eligible to act at the next substage;

η2. If $s+1$ is β-expansionary and Δ_η^A is incorrect at (the least) y, then remove the numbers, z say, enumerated into C between $s+1$ and the stage at which $\Delta_\eta^A(y)[s]$ was defined, put $\gamma(k(\eta))[s]$ into D. If z is an ω_1-marker, then put $z-1$ into D. Let x be the number such that z is in the interval of $\omega_1(x)$. If x is currently in E, then take x out, and if x is currently not in E, then put x into E. Also put $\delta_\beta(y)[s]$ into A to undefine all those $\Delta_{\eta'}^A(y)$ defined after stage $\delta_\eta(y)[s]$ *(Remember that $\delta_\eta(y)[s]$ is a stage at which $\Delta_\eta^A(y)[s]$ is defined).*

Define $d(\eta)\downarrow$, indicating that η has executed the isolation argument, and define $\sigma_{s+1} = \eta$, initialize all $\zeta \not\leq \sigma_{s+1}$, and go to the next stage.

η3. If $s+1$ is not η-expansionary, $d(\eta)\downarrow$, then let $\eta^\frown d$ be eligible to act at the next substage;

η4. Otherwise, let $\eta^\frown 1$ be eligible to act at the next substage.

`Case 4.` $\xi = \tau$ is an $\mathcal{R}$-strategy.

τ1. If $s+1$ is a τ-expansionary stage, then let $\tau^\frown 0$ be eligible to act at the next substage;

τ2. Otherwise, let $\tau^\frown 1$ be eligible to act at the next substage.

This completes the construction.

3.4 Verification

In this section, we verify that our construction satisfies all the requirements. First we have:

Lemma 3.1. *For any x, s,*

(1) If $\Gamma^{C,D}(x)[s]\downarrow$, then $\gamma(x)[s] \notin C_s \cup D_s$;

(2) If $\Gamma^{C,D}(x+1)\downarrow$, then $\Gamma^{C,D}(x)\downarrow$ and $\gamma(x)[s] < \gamma(x+1)[s]$;

(3) If $s < v$, $\Gamma^{C,D}(x)[s]\downarrow$, $C_v \upharpoonright (\gamma(x)[s]+1) = C_s \upharpoonright (\gamma(x)[s]+1)$ and $D_v \upharpoonright (\gamma(x)[s]+1) = D_s \upharpoonright (\gamma(x)[s]+1)$, then $\Gamma^{C,D}(x)[v]\downarrow$, and $\gamma(x)[v] = \gamma(x)[s]$;

(4) If $\Gamma^{C,D}(x)$ is undefined at stage $s+1$, then $C_{s+1} \upharpoonright (\gamma(x)[s]+1) \neq C_s \upharpoonright (\gamma(x)[s]+1)$ or $D_{s+1} \upharpoonright (\gamma(x)[s]+1) \neq D_s \upharpoonright (\gamma(x)[s]+1)$. □

Define $f = \liminf_s \sigma_s$. f is the true path of the construction.

Lemma 3.2. *For $\xi \subset f$,*

(1) ξ is initialized or reset only finitely often;

(2) ξ acts only finitely often.

Proof. We prove the lemma by induction.

Let ξ^- be the immediate predecessor of ξ. By the induction hypotheses, (a) ξ^- cannot be initialized or reset afterwards, (b) ξ^- does not act afterwards.

Since ξ is on f, there is some stage $s_1 \geq s_0$ such that $\forall s \geq s_1(\sigma_s \geq \xi)$. By the choice of s_0, and the fact that ξ acts only when ξ is visited, ξ cannot be initialized after s_1. Furthermore, if ξ is an $\mathcal{M}$ or an $\mathcal{N}$-strategy, let $k(\xi)$ be defined at stage $s_2 \geq s_1$. Then $k(\xi)$ cannot be cancelled later and ξ can be reset only finitely often. (1) holds.

For (2), there are four cases:

(i) ξ is a $\mathcal{P}$ or a $\mathcal{Q}$-strategy.

Suppose that at stage $s_3 > s_1$, ξ chooses a fresh number $x(\xi)$ as an attacker for C, and that at stage $s_4 > s_3$ at which ξ is visited again, $\Phi_\xi^A(x(\xi))[s_4] \downarrow= 0$. If no such a stage exists, then ξ is obviously satisfied. By the construction, $x(\xi)$ is put into C. Since all the strategies with lower priority are initialized at this stage, and they cannot put small numbers into A to change the computation $\Phi_\xi^A(x(\xi))$. By the choice of s_1, no strategies with higher priority will act after this stage. Therefore, $\Phi_\xi^A(x(\xi))$ is preserved forever. ξ is satisfied and does not act anymore. (2) holds.

(ii) ξ is a $\mathcal{T}$-strategy.

It is quite similar to (i), except that $x(\xi)$ as an attacker for E.

Note that when ξ puts $x(\xi)$ into E, ξ also puts $\omega_1(x(\xi))$ and $\omega_2(x(\xi))$ into C and D respectively, to undefine $\Omega_1^C(x(\xi))$ and $\Omega_2^D(x(\xi))$. So generally, ξ's action can lead to $\mathcal{M}$ or $\mathcal{N}$-strategies with higher priority to perform the isolation argument, which means that these strategies can remove $x(\xi)$ from E or enumerate $x(\xi)$ into E again. If this happens, then ξ will be initialized. By the choice of s_1, we know that after this stage, if ξ enumerates $x(\xi)$ into E, then no other strategy will take it out, and hence, ξ is satisfied forever, which means that ξ does no more actions. (2) holds.

(iii) ξ is an $\mathcal{M}$ or an $\mathcal{N}$-strategy.

W.o.l.g., suppose that ξ acts after stage s_2. Then some disagreement between Δ_ξ^A and W_ξ, y say, appears at stage s_ξ say. Let $\Delta_\xi^A(y)[s_2]$ be defined at stage $s' < s_2$. By the construction, ξ performs the isolation argument by taking out all numbers enumerated into C between stages s' and s_ξ and the computation $\Phi_\xi^{A,C}(y)$ is recovered to $\Phi_\xi^{A,C}(y)[s']$. By the choice of s_1, no strategy on the tree will change this computation after stage s_ξ.

Now we consider whether this computation can be injured by the $\mathcal{G}$, $\mathcal{S}$-strategies. Note that at stage s_ξ, while we remove numbers from C, we also enumerate $\gamma(k(\xi))[s_\xi]$ into C, to lift all $\gamma(n)$, $n \geq k(\xi)$. In particular, when $\Gamma^{C,D}(n), n \geq k(\xi)$, is redefined, $\gamma(n)$ will be defined as a big number, and the enumeration of these newly defined $\gamma(n)$ into C will not injure the computation $\Phi_\xi^{A,C}(y)$. By the choice of s_2, no $y < k(\xi)$ will enter K afterwards, and hence no $\gamma(y)$ with $y < k(\xi)$ can enter C later. This ensures that $\Phi_\xi^{A,C}(y)$ cannot be injure by the $\mathcal{G}$-strategy.

Since whenever a $\mathcal{T}$-strategy or an $\mathcal{M}$ or $\mathcal{N}$-strategy acts, it already take the correctness of Ω_1 and Ω_2 into account. That is, these strategies never violate the correctness of these two functionals. Therefore, the $\mathcal{S}$-strategy itself does not need to rectify the errors during the whole construction.

We can conclude now that no strategy can injure the computation $\Phi_\xi^{A,C}(y)$, and this computation will be preserved forever. Thus, we have:

$$\Phi_\xi^{A,C}(y) = \Phi_\xi^{A,C}(y)[s'] = W_{\xi,s'}(y) = 0 \neq 1 = W_{\xi,s_\xi}(y) = W_\xi(y).$$

ξ is satisfied and does not act anymore. (2) holds.

(iv) ξ is an $\mathcal{R}$-strategy.

ξ does no action after stage s_2. (2) follows immediately.

□

By Lemma 3.2, we have the following lemma immediately.

Lemma 3.3. *For any $\xi \subset f$, there is some $\mathcal{O}$ such that $\xi^\frown\mathcal{O} \subset f$. Thus, $|f| = \infty$.*
□

Lemma 3.4. *For any $\xi \subset f$,*

(1) If ξ is a $\mathcal{P}$-strategy, then $\mathcal{P}$ is satisfied.

(2) If ξ is a $\mathcal{Q}$-strategy, then $\mathcal{Q}$ is satisfied.

(3) If ξ is a $\mathcal{M}$-strategy, then $\mathcal{M}$ is satisfied.

(4) If ξ is a $\mathcal{N}$-strategy, then $\mathcal{N}$ is satisfied.

(5) If ξ is an $\mathcal{R}$-strategy, then $\mathcal{R}$ is satisfied.

Proof. (1) and (2) follow the proof of (i) in Lemma 3.2 immediately.

(3) Suppose that $\Phi_\xi^{A,C} = W_\xi$. By the proof of (ii) in Lemma 3.2, ξ has no chance to initiate the isolation argument, because otherwise, a disagreement between W_ξ and $\Phi_\xi^{A,C}$ would be created and be preserved forever, making $\Phi_\xi^{A,C} \neq W_\xi$, a contradiction. Thus, we only need to prove that Δ_ξ^A is total.

Fix x. Let $\Delta_\xi^A(x)$ be defined at stage s_0. Then $\delta_\xi^A(x)[s_0] = s_0$. W.o.l.g., suppose that at stage $s_1 > s_0$, some $\xi' > \xi$ executes the isolation strategy, and $\Delta_\xi^A(x)$ is declared to be undefined. Then at the next ξ-expansionary stage s_2, $\Delta_\xi^A(x)$ is redefined with use $\delta_\xi^A(x)[s_2] = s_2$. Since all the strategies $> \xi'$ are initialized at stage s_1 and cannot redefine $\Delta_\xi^A(x)[s_2]$, because all the numbers enumerated into A by these strategies will be bigger than s_2. Since only finitely many nodes between $\xi^\frown 0$ and ξ' have been visited before stage s_2, and only these strategies' action can undefine the newly defined $\Delta_\xi^A(x)$, there is a large enough stage s_x at which $\Delta_\xi^A(x)$ is defined and this definition cannot be destroyed afterwards. Then $\Delta_\xi^A(x) \downarrow = \Delta_\xi^A(x)[s_x]$.

Therefore, Δ_ξ^A is total and computes W_ξ correctly. ξ is satisfied.

(4) ξ is an $\mathcal{N}$-strategy. The proof that ξ is satisfied at ξ is the same as the proof for $\mathcal{M}$-strategies.

(5) Assume that $\Phi_\xi^A = \Phi_\xi^B = g$ is total. Since ξ is on the true path, we know from Lemma 3.2 that ξ can be initialized only finitely often. Let s_0 be the least stage such that ξ cannot be initialized afterwards(0, if no such a stage exists). By assumption, there are infinitely many ξ-expansionary stages bigger than s_0. We can compute g effectively as follows.

Given $x > s_0$, find the least ξ-expansionary stages s_1 say, $s_1 > s_0$ and $\ell_R(\xi, s_1) > x$. Then $g(x) = \Phi^A_\xi(x)[s_1] = \Phi^B_\xi(x)[s_1]$ by the argument in the $\mathcal{R}$-strategy. Therefore, g is computable. □

Lemma 3.5. $\Gamma^{C,D} = K$. *Hence, $\mathcal{G}$ is satisfied.*

Proof. By lemma 1, $\Gamma^{C,D}$ is a p.c. functional. By actions during stages $3k+1$, if $\Gamma^{C,D}$ is total, then $\Gamma^{C,D} = K$.

Fix k. Let η be an $\mathcal{M}$ or an $\mathcal{N}$-strategy on the true path f with $|\eta| \geq k$. Then η can only be visited after stage k. W.o.l.g., suppose that η is an $\mathcal{M}$-strategy. Since η is on the true path, by Lemma 3.2, there is some (least) stage s_η such that η cannot be initialized or reset afterwards. Let $s' \geq s_\eta$ be the stage at which $k(\eta)$ is defined. Then $k(\eta) > k$ and $k(\eta)$ cannot be cancelled later. We may assume that η acts at stage $s'' > s'$. Then at this stage, $\gamma(k(\eta))[s']$ is put into D, undefining all $\gamma(n)$ with $n \geq k(\eta)$. Since all lower priority strategies are initialized at stage s', $C \upharpoonright \gamma(k(\eta))[s']$ (hence $C \upharpoonright \gamma(k)[s']$) remains unchanged. Also by the choice of s_η, η cannot be reset again, no γ-markers less than $\gamma(k(\eta))[s']$ can enter D later. Thus,

$$C \upharpoonright \gamma(k)[s'] = C_{s'} \upharpoonright \gamma(k)[s'] \ \text{ and } \ D \upharpoonright \gamma(k)[s'] = D_{s'} \upharpoonright \gamma(k)[s'].$$

Therefore, $\Gamma^{C,D}(k) \downarrow = \Gamma^{C,D}(k)[s']$. □

Lemma 3.6. $E = \Omega^C_1 = \Omega^D_2$. *Hence, $\mathcal{S}$ is satisfied.*

Proof. Fix x.

If x is not an attacker of a $\mathcal{T}$-strategy, then $\Omega^C_1(x)$ and $\Omega^D_2(x)$ are defined as $E(x) = 0$. This definition keeps on, since no strategy puts x into E, $\Omega^C_1(x) = \Omega^D_2(x) = 0 = E(x)$ holds.

Now suppose that x is an attacker of a $\mathcal{T}$-strategy β, then $\Omega^C_1(x)$ and $\Omega^D_2(x)$ are defined as $E(x)$, at stage s_0 say. After this, if x is put into E, then at the same stage, $\omega_1(x), \omega_2(x)$ (they are actually the same) are enumerated into C and D respectively to undefine $\Omega^C_1(x)$ and $\Omega^D_2(x)$. This action may lead to an $\mathcal{M}$-strategy (or an $\mathcal{N}$-strategy), $\eta \subset \beta$ say, to perform the isolation strategy, at stage s_1 say. If so, then η will take $\omega_1(x)$ out of C, which forces us to take x out of E, since $C \upharpoonright (\omega_1(x) + 1)$

is recovered to C_{s_0}. Consequently, to ensure that $\Omega_2^D(x)$ computes $E(x)$ correctly, η puts $\omega_2(x) - 1$ into D to undefine $\Omega_2^D(x)$. Thus, $\Omega_1^C(x)$ is defined now and equals to $E(x)$ and $\Omega_2^D(x)$ is undefined.

When $\Omega_2^D(x)$ is defined again, it is defined as $E(x) = 0$, with the use the same as the original $\omega_2(x)$, even though we will use numbers less than $\omega_2(x)$ to undefine it later if necessary. Remember that when we define $\omega_2(x)$ for the first time, we have save an interval of numbers for this purpose.

Since at stage s_1, $\omega_2(x) - 1$ is enumerated into D, and such an enumeration can lead to an $\mathcal{N}$-strategy $\eta' \subset \eta$ to perform the isolation strategy. If so, then $\omega_2(x) - 1$ is removed from D. If so, then we need to put x back into E, and correspondingly, we need to put $\omega_1(x) - 2$ into C, to undefine $\Omega_1^C(x)$. Again, when $\Omega_1^C(x)$ is defined, we define it as $E(x)$, with the use the same as the original $\omega_2(x)$.

Thus, after x is put into E, the process described above can happen at most $|\beta|$ many times, since only those $\mathcal{M}$ and $\mathcal{N}$-strategies above β can perform like this, changing the status of $E(x)$. *Hence, if we let $x > |\beta|$, then $E(x)$ can change at most x many times, and E is ω-c.e.*

Let s_x be the last stage we change $E(x)$, then between stages s_0 and s_1, at any stage, either one of $\Omega_1^C(x)$ and $\Omega_2^D(x)$ is defined and equals to $E(x)$, and the other one is undefined, or both of them are defined and equal to $E(x)$. After s_x, once we define $\Omega_1^C(x)$ or $\Omega_2^D(x)$ again, both of them will be defined and equal to $E(x)$. After this, no strategy will change $E(x)$, and $\Omega_1^C(x)$ and $\Omega_2^D(x)$ will be defined forever, with $\Omega_1^C(x) = \Omega_2^D(x) = E(x)$. Therefore, $\Omega_1^C(x)$ and $\Omega_2^D(x)$ converge and compute $E(x)$ correctly. □

This completes the proof of Theorem 3. ◇◇

Chapter 4

INFIMA OF D.C.E. DEGREES IN ERSHOV HIERARCHY

4.1 Introduction

In this chapter, we consider the dense property of infima of the d.c.e. degrees. Lachlan observed that the infimum of two c.e. degrees considered in the c.e. degrees coincides with the one considered in the Δ_2^0 degrees. This result cannot be extended to the d.c.e. degrees. Kaddah proved in [44] that there are *d.c.e.* degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and a 3-*c.e.* degree $\mathbf{x}$ such that $\mathbf{a}$ is the infimum of $\mathbf{b}, \mathbf{c}$ in the d.c.e. degrees, but not in the 3-c.e. degrees, as $\mathbf{a} < \mathbf{x} < \mathbf{b}, \mathbf{c}$. Here, we extend Kaddah's result by showing that

Theorem 4 (Liu, Wang and Wu [54]) *Given c.e. degrees* $\mathbf{u} < \mathbf{v}$*, there are d.c.e. degrees* $\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}$ *and a* 3*-c.e. degree* $\mathbf{x}$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{a} < \mathbf{x} < \mathbf{b_1}, \mathbf{b_2}$*, and at the same time,* $\mathbf{b_1}$ *and* $\mathbf{b_2}$ *have infimum* $\mathbf{a}$ *in the d.c.e. degrees.*

4.2 Requirements and basic strategies

In this section, we list the requirements needed to prove Theorem 4, and describe the strategy for satisfying each requirement. Given U and V with $U <_T V$, we will construct three *d.c.e.* sets A, B_1, B_2, a 3-*c.e.* set X and auxiliary functionals Γ_1, Γ_2

and Δ_j satisfying the following requirements:

$\mathcal{G}$**:** $A, B_1, B_2, X \leq_T V$;

$\mathcal{R}$**:** $X = \Gamma_1^{B_1,A,U} = \Gamma_2^{B_2,A,U}$;

$\mathcal{P}_n$**:** $X \neq \Phi_n^{A,U}$;

$\mathcal{N}_n$**:** $\Phi_e^{B_1,A,U} = \Phi_e^{B_2,A,U} = D_j \Rightarrow D_j = \Delta_j^{A,U}$,

where $n = \langle e, j \rangle$ and $\{D_j : j \in \omega\}$ is some standard listing of all *d.c.e.* sets.

Let $\mathbf{a}, \mathbf{x}, \mathbf{b_1}, \mathbf{b_2}$ be the degrees of $A \oplus U, X \oplus A \oplus U, B_1 \oplus A \oplus U, B_2 \oplus A \oplus U$ respectively. By the requirements $\mathcal{G}$ and $\mathcal{R}$, $\mathbf{u} \leq \mathbf{a}, \mathbf{x}, \mathbf{b_1}, \mathbf{b_2} \leq \mathbf{v}$. By the $\mathcal{P}$-requirements, $\mathbf{a} < \mathbf{x}$. The $\mathcal{N}$-requirements ensure that $\mathbf{a} = \mathbf{b_1} \wedge \mathbf{b_2}$ in the d.c.e. degrees. Here we only require that these degrees are between $\mathbf{u}$ and $\mathbf{v}$. Sacks' density theorem ensures that these degrees are strictly between $\mathbf{u}$ and $\mathbf{v}$.

4.2.1 The $\mathcal{G}$ and $\mathcal{R}$-strategies

The $\mathcal{G}$ and $\mathcal{R}$ requirements are both global. To satisfy $\mathcal{G}$, we apply the delayed permitting argument, which was first introduced by Sacks in [73] to prove the Theorem 1.3. For the $\mathcal{R}$-requirement, we apply the usual coding strategy. That is, we enumerate two functional axioms Γ_i with $i = 1, 2$ following the rules below:

(1) When $\Gamma_i^{B_i,A,U}(x)$ is first defined at a stage s, its use $\gamma_i(x)[s]$ is selected as a fresh number.

(2) If x enters or exits X at stage t after $\Gamma_i^{B_i,A,U}(x)$ is defined, then $\Gamma_i^{B_i,A,U}(x)[t]$ must be undefined or recovered to ensure that it is finally defined agreeing with $X(x)$. In our construction, we put or, extract a number less than or equal to $\gamma_i(x)[t]$ from $B_i \cup A$.

(3) For every x, there is a stage s_x such that after s_x, $\Gamma_i^{B_i,A,U}(x)$ can never be undefined.

(4) $\gamma_i(x)$ is increasing with respect to x.

4.2.2 A $\mathcal{P}$-strategy

The basic idea of satisfying a $\mathcal{P}$-requirement is the standard Friedberg-Muchnik argument. That is,

(1) Pick a witness x for $\mathcal{P}$;

(2) Wait for $\Phi^{A,U}(x) \downarrow= 0$;

(3) Put x into X and impose a restraint on $A \upharpoonright \varphi(x)$.

The problem is that after $\Phi^{A,U}(x) \downarrow= 0$, the computation $\Phi^{A,U}(x)$ may be destroyed by $U \upharpoonright \varphi(x)$ even though we put restraint $A \upharpoonright \varphi(x)$. To get out of this dilemma, we will impose "indirect" restraint on U by threatening $V \leq_T U$ via Ψ.

Let α be a $\mathcal{P}$-strategy. α will run cycles j for $j \in \omega$. α starts cycle 0 first. Each cycle j can start only cycle $j+1$, however stop any cycle j' with $j' > j$. All the cycles of α define a functional Ψ_α jointly. The cycle j is responsible for the definition of $\Psi_\alpha^U(j)$.

Given $j \in \omega$, α runs cycle j as follows:

(1) Pick x_j as a fresh number.

(2) Wait for a stage s_0 such that $\Phi^{A,U}(x_j)[s_0] \downarrow= 0$.

(3) Preserve $A \upharpoonright \varphi(x_j)[s_0]$ from other strategies.

(4) Set $\Psi_\alpha^U(j)[s_0] = V(j)[s_0]$ with use $\psi_\alpha(j) = \varphi(x_j)[s_0]$ and start cycle $j+1$ simultaneously.

(5) Wait for $U \upharpoonright \varphi(x_j)[s_0]$ or $V(j)$ to change.

   (a) If $U \upharpoonright \varphi(x_j)[s_0]$ changes first, then cancel all cycles $j' > j$ and drop the A-restraint of cycle j to 0. Go back to step 2.

   (b) If $V(j)$ changes first, then stop cycles $j' > j$, and go to step 6.

(6) Put x_j into X and wait for $U \upharpoonright \varphi(x_j)[s_0]$ to change.

(7) Define $\Psi^U_\alpha(j) = V(j) = 1$ with use 0, and start cycle $j+1$.

α has two sorts of outcomes:

(j, f) : There is a stage s after which no new cycle runs.

(*So some cycle j_0 waits forever at step 2 or 6. If α waits forever at step 2 then $\Phi^{A,U}(x_{j_0})\downarrow= 0$ is not true. If α waits for ever at step 6 then $\Phi^{A,U}(x_{j_0})\downarrow= 0$ and $x_{j_0} \in X$. In any case, x_{j_0} is a witness for the requirement $\mathcal{P}$.*)

(j, ∞) : Some cycle j_0 runs infinitely often, but no cycle $j' < j$ does so.

(*It must be true that cycle j_0 goes from step 5 to 2 infinitely often. Thus $\Phi^{A,U}(x_{j_0})$ diverges. So x_{j_0} is clearly a witness for the requirement $\mathcal{P}$.*)

Note that it is impossible that, there are stages s_j for all $j \in \omega$ such that no cycle j runs after stage s_j but there are infinitely many stages at which some cycle runs. If so, then for each j, cycle j will stay at step 5 or 7 eventually, and then all these cycles will define Ψ^U totally and $\Psi^U = V$ (so $V \leq_T U$), a contradiction. (*By the $\mathcal{P}$-strategy, whenever a cycle j is started, any previous version of it has already been cancelled by a U-change. So Ψ^U_α is well-defined.*)

Now, we consider the interaction between strategies $\mathcal{P}$ and $\mathcal{G}(\mathcal{R})$. When α puts a number x_j into X at step 6, if $\Gamma_i^{B_i,A,U}(x_j)$ is defined then we must make it undefined. To do this, we will put its use $\gamma_i(x_j)$ into B_i. But for $B_i \leq_T V$, we have to get a V-permitting. In this construction the V-permitting is realized via a $V(j)$-change. So it suffices to modify step 6 as follows:

(6′) Put x_j into X, $\gamma_i(x_j)$ into B_i and wait for $U \upharpoonright \varphi(x_j)[s_0]$ to change.

For the convenience of description, we introduce some notions here. Say that a *j-cycle acts* if it chooses a fresh number x_j as its attacker at step 1 or, it changes the value of $X(x_j)$ by enumerating x_j into X at step 6′. We also say that j-cycle *is active* at stage s if at this stage, when α is visited, α is running j-cycle, except the situation that j-cycle is just started at stage s.

4.2.3 An $\mathcal{N}$-Strategy

An $\mathcal{N}_e$-strategy, β say, is devoted to the construction of a partial functional Δ_β such that if $\Phi_e^{B_1,A,U} = \Phi_e^{B_2,A,U} = D_j$ then $\Delta_\beta^{A,U}$ is well defined and computes D_j correctly. For simplicity, we will omit the subscript in the basic strategy, and it will be indicated if need.

First define the agreement function as follows:

$$l(\beta, s) = \max\{y < s : \Phi^{B_1,A,U} \upharpoonright y[s] \downarrow = \Phi^{B_2,A,U} \upharpoonright y[s] \downarrow = D_s \upharpoonright y)\},$$

$$m(\beta, s) = \max\{0, l(\beta, t) : t < s \ \& \ t \text{ is a } \beta\text{-stage}\}.$$

Say a stage s is β-expansionary if $s = 0$ or s is an β-stage such that $l(\beta, s) > m(\beta, s)$.

If neither the U-changes nor the V-permission is involved, then the basic strategy for this requirement is exactly the same as the one introduced by Kaddah [44]: After $\Delta^{A,U}(z)$ was defined, we allow the computations $\Phi^{B_i,A,U}(z)$'s to be destroyed simultaneously (by $\mathcal{P}$-strategies) instead of preserving one of them as in Lachlan's minimal pair construction. As a consequence, as we are constructing B_i d.c.e., we are allowed to remove numbers from one of them to recover a computation to a previous one, and force a disagreement between D and $\Phi^{B_i,A,U}$ for some $i \in \{1, 2\}$ at an argument z.

(1) Define $\Delta^{A,U}(z)[s_0] = D_{s_0}(z)$ with $\delta(z) = s_0$ at some β-expansionary stage s_0. (We assume that $D_{s_0}(z) = 0$ here, mainly because it is the most complicated situation.)

(2) Some $\mathcal{P}_e$-strategy with lower priority enumerates a small number x into X_{s_1} at stage $s_1 > s_0$, and for the sake of the global requirement $\mathcal{R}$, we put $\gamma_1(x)[s_1]$ into B_{1,s_1} and $\gamma_2(x)[s_1]$ into B_{2,s_1}. The computations $\Phi^{B_i,A,U}(z)[s_0]$ could *be injured* by these enumerations.

(3) At stage $s_2 > s_1$, z enters D_{s_2}.

(4) Then at some β-expansionary stage $s_3 > s_2$, we force that $D(z) \neq \Phi^{B_1,A,U}(z)$ by extracting $\gamma_1(x)[s_1]$ from B_{1,s_3} to recover the computation $\Phi^{B_1,A,U}(z)$ to

$\Phi^{B_1,A,U}(z)[s_0]$. To make sure that Γ_1 and Γ_2 well-defined and compute X correctly, we need to extract x from X, and put a number, s_1, into $B_2 \oplus A$, to undefine $\Gamma_2^{B_2,A,U}(x)$. *We put s_1, instead of the current $\gamma_2(x)[s_3]$ into A, mainly because we need to ensure that β's action is consistent with those higher $\mathcal{N}$-strategies.*

(5) At stage $s_4 > s_3$, z goes out of D_{s_4}. *Since D is d.c.e., z cannot enter D later.*

(6) At a β-expansionary stage $s_5 > s_4$, $\Phi^{B_1,A,U}(z)[s_5] = \Phi^{B_2,A,U}(z)[s_5] = D_{s_5}(z) = 0$ again. Now, we are ready to force that $\Phi^{B_2,A,U}(z) \neq D(z)$ by extracting s_1 from A (to recover the computation $\Phi^{B_2,A,U}(z)$ to $\Phi^{B_2,A,U}(z)[s_3]$) and enumerating x into X. To correct $\Gamma_1^{B_1,A,U}(x)$, we put a number less than $\gamma_1(x)[s_1]$, $\gamma_1(x)[s_1]-1$ say, into B_{1,s_5}.

Now we consider the U and V-changes involved in such a process. Note that step 4 and 6 assume that $U \upharpoonright \varphi(B_1, A, U; z)[s_0]$ and $U \upharpoonright \varphi(B_1, A, U; z)[s_3]$ don't change after stage s_0 and s_3 respectively. If $U \upharpoonright \varphi(B_1, A, U; z)[s_0]$ changes then it can correct $\Delta^{A,U}(z)[s_0]$. Consequently, this U-change allow us to correct $\Delta^{A,U}(z)[s_0]$ (this U-change may damage our plan of making a disagreement at z). If the change happens after step 4, we will deal with this by threatening $V \leq_L U$ via a functional Θ. That is, we will make infinitely many attempts to satisfy $\mathcal{N}$ as above by an infinite sequence of cycles.

Every cycle k will run as above, in addition, insert a step before step 4:

(4°) Set $\Theta^U(l) = V(l)[s_3]$ with use $\theta(l) = \delta(z)[s_3]$. Start $(l+1)$-cycle simultaneously, wait for $V(l)$-change (i.e. V-permitting), then stop the cycles $l' > l$ and go to step 5.

Once some cycle finds a $U \upharpoonright \theta(l)$-change after stage s_3, we will stop all cycles $l' > l$ and go back to step 1 as the $U \upharpoonright \theta(l)$-change can correct $\Delta^{A,U}(z)[s_0]$.

A similar modification applies to step 6. We attach an additional step 6° before step 6 so as to run the disagreement strategy and pass through another V-permitting argument.

To combine two disagreement strategies, we need two V-permissions, and as in [17], we arrange the basic $\mathcal{N}$-strategy in $\omega \times \omega$ many cycles (k, l) where $k, l \in \omega$. Their priority is arranged by the lexicographical ordering. $(0, 0)$-cycle starts first, and each (k, l)-cycle can start cycles $(k, l+1)$ or $(k+1, 0)$ and stop, or cancel cycles (k', l') for $(k, l) < (k', l')$. Each (k, l)-cycle can define $\Theta_k^U(l)$ and $\Xi^U(k)$. At each stage, only one cycle can do so.

(k, l)-cycle runs as follows:

(1) Wait for a β-expansionary stage, s say.

(2) If $\Delta^{A,U}(z)[s] = D_s(z)$ for all z such that $\Delta^{A,U}(z)[s]\downarrow$, then go to (3). Otherwise, let z be the least one such that $\Delta^{A,U}(z)[s] \downarrow\neq D_s(z)$, and then go to step 4.

(3) Define $\Delta^{A,U}(z)[s] = D_s(z)$ for all $z < l(\beta, s)$ that $\Delta^{A,U}(z)[s]\uparrow$ with $\delta(z) = s$, then go back to step 1.

(4) Restrain $A \upharpoonright \delta(z)[s]$, $B_1 \upharpoonright \delta(z)[s]$ and $B_2 \upharpoonright \delta(z)[s]$ from other strategies. Go to step 5.

(5) There are two cases:

Case 1 If $D_s(z) = 1$, then set $\Theta_k^U(l) = V_s(l)$ with $\theta_k(l)[s] = \delta(z)[s]$ ($\delta(z)[s] < s$). Start $(k, l+1)$-cycle simultaneously. Go to step 6.

Case 2 If $D_s(z) = 0$, then set $\Xi^U(k) = V_s(k)$ with $\xi(k)[s] = \delta(z)[s]$. Start $(k+1, 0)$-cycle simultaneously and go to step 7.

(6) Wait for $U \upharpoonright \theta_k(l)[s]$ or $V(l)$ to change.

(a) If $U \upharpoonright \theta_k(l)[s]$ changes first then cancel cycles $(k', l') > (k, l)$, drop the A, B_1, B_2-restraints of (k, l)-cycle to 0, undefine $\Delta^{A,U}(z)[s]$ and go back to step 1.

(b) If $V(l)$ changes first at stage $t > s$ then we just remove $\gamma_1(x)[s_0]$ from B_1 and x from X_t, put s_0 into A, where $s_0 < s$ is the stage at which x is enumerated into X. Now stop cycles $(k', l') > (k, l)$ and go back to step 1.

(7) Wait for $U \upharpoonright \xi(k)[s]$ or $V(k)$ to change.

(I) If $U \upharpoonright \xi(k)[s]$ changes first then cancel cycles $(k', l') > (k, l)$, drop the A, B_1, B_2-restraints of (k, l)-cycle in step 4 to 0, undefine $\Delta^{A,U}(z)[s]$ and go back to step 1.

(II) If $V(k)$ changes first at stage $t > s$ then we just remove s_0 from A and, put x into X and $\gamma_1(x)[s_0] - 1$ into B_1. Now stop cycles $(k', l') \geq (k+1, 0)$ and go to step 8.

(8) Wait for $U \upharpoonright \xi(k)[s]$ to change. If this change happens then we take actions as follows: Cancel cycles $(k', l') > (k, l)$, drop the A, B_1, B_2-restraints of (k, l)-cycle in step 4 to 0, undefine $\Delta^{A,U}(z)[s]$ and go back to step 1. *Note that such a U-change also undefine $\Xi^U(k)$, and we select to run cycle (k, l), instead of starting cycle $(k+1, 0)$, even though we can select anyone. We can do this, as $\Xi^U(k)$ is now undefined, as we can define it later, when needed. Furthermore, it gives a chance to correct $\Delta^{A,U}(z)[s]$.*

β has four possible outcomes:

(1) Some cycle (k, l) (the leftmost one) runs infinitely often, then $\Delta^{A,U}$ is totally defined and computes D correctly.

(2) Some cycle (k, l) waits at step 1 eventually, then a disagreement appears between $\Phi^{B_1,A,U}$ and D, or between $\Phi^{B_2,A,U}$ and D.

(3) There is a k such that for every l there is a stage s_l such that no cycle (k, l) acts after stage s_l. Then every cycle wait at step 6 for ever. So Θ_k^U computes V correctly, a contradiction.

(4) For every k there is a stage s_k such that no cycle (k, l) acts after stage s_k, then for every k there is a cycle (k, l_k) that waits at step 7 for ever. So Ξ^U computes V correctly, a contradiction.

4.3 Construction

First of all, the priority of the requirements are arranged as follows:

$$\mathcal{G} < \mathcal{R} < \mathcal{N}_0 < \mathcal{P}_0 < \mathcal{N}_1 < \mathcal{P}_1 < \cdots < \mathcal{N}_n < \mathcal{P}_n < \cdots ,$$

where each $\mathcal{X} < \mathcal{Y}$ mean that, as usual, $\mathcal{X}$ has higher priority than $\mathcal{Y}$.

Each $\mathcal{N}$-strategy has outcomes $(k,l), k,l \in \omega$, and f in order type ω^2+1. Their priority is arranged as follows: the cycles $(k,l), k,l \in \omega$, are in lexicographical ordering which denote the outcome $(O1)$ in the last section, and f has the lowest priority which denotes finitely many expansionary stages.

Each $\mathcal{P}$-strategy has outcomes (j,f), (j,∞) for $j \in \omega$ in order type ω. Their priority is arranged as follows:

$$(0,\infty) <_L (0,f) <_L \cdots <_L (n,\infty) <_L (n,f) < \cdots .$$

The priority tree, T, is constructed computably by the outcomes of the strategies corresponding to the requirements and grows downwards. The requirements $\mathcal{G}$ and $\mathcal{R}$ are both global, and hence their outcomes are not on T.

The full construction is given as follows. The construction will proceed by stages. At odd stages, we define Γ_i's; at even stages, approximate the true path. For the sake of clarifying what a number enters B_i, we require that the $\gamma_i(-)$-uses are defined by odd numbers.

Full Construction

Stage 0: Initialize all nodes on T, and let $A = B_1 = B_2 = X = \Gamma_1 = \Gamma_2 = \emptyset$. Let $\sigma_0 = \lambda$.

Stage $s = 2n+1 > 0$: Define $\Gamma_1^{B_1,A,U}(x) = \Gamma_2^{B_2,A,U}(x) = X_s(x)$ for least $x < s$ for which Γ's are not defined, with fresh uses $\gamma_1(x)[s]$ and $\gamma_2(x)[s]$.

Stage $s = 2n > 0$:

Substage 0: Let $\sigma_s(0) = \lambda$ the root node.

Substage t: Given $\zeta = \sigma_s \upharpoonright t$. First initialize all the nodes $>_L \zeta$. If $t = s$ then define $\sigma_s = \zeta$ and initialize all the nodes with lower priority than σ_s.

If $t < s$, take action for ζ and define $\sigma_s(t+1)$ depending on which requirement ζ works for.

Case 1 If $\zeta = \alpha$ is a $\mathcal{P}$-strategy, there are three subcases.

Subcase (α1) If α has no cycle started, then start cycle 0 and choose a fresh number $x_{\zeta,0}$ as its attacker. Define $\sigma_s = \zeta^\frown(0, f)$, initialize all nodes with lower priority.

Subcase (α2) If α1 fails, let j be the largest active cycle at the last ζ-stage if any. Now, implement the delayed permitting (if any) as follows:

(α2.1) If U has a change below the restrain of a cycle $j' \leq j$, without loss of generality, assume j' to be the least one, then define $\sigma_s(t+1)$ as $\alpha^\frown(j', \infty)$ and then go to the next substage, if cycle j' has not received the j'-permission so far; and define $\sigma_s(t+1)$ as $\alpha^\frown(j'+1, f)$, if cycle j' received a j'-permission before. In the second case, redefine $\psi_\alpha(j')$ as $\psi_\alpha(j'-1)[s]$ since j' is in V_s and define $x_{\alpha,j'+1}$ as a new number, now define $\sigma_s = \alpha^\frown(j'+1, f)$, initialize all the nodes with lower priority than σ_s and go to the next stage.

(α2.2) If α2.1 fails and V has a change on a number $j' \leq j$ (without loss of generality, assume j' to be the least one) between the last α-stage and stage $s+1$, then let cycle j' act at this stage. So j'-cycle will enumerate $x_{\alpha,j'}$ into X and $\gamma_i(x_{\alpha,j'})$ into B_i. Say that j'-cycle of α receives j'-permission at stage $s+1$. Initialize all the nodes with priority lower than or equal to $\alpha^\frown(j', f)$ and go to the next stage. Let $\sigma_s = \alpha^\frown(j', f)$. Here we say that α is satisfied via j'-cycle till U changes below the corresponding use $\psi_\alpha(j')$.

Subcase (α3) If Subcases α1 and α2 fail, then take actions as follows:

If α is at j-cycle, not satisfied yet, and $\Phi^{A,U}_{e(\alpha)}(x_{\alpha,j}) \downarrow = 0$, then set $\Psi^U_\alpha(j) = V(j)$. The use $\psi_\alpha(x_{\alpha,j})$ is defined as $\varphi_{e(\alpha)}(x_{\alpha,j})[s]$, if $j \notin V_s$, and defined as

$\psi_\alpha(j-1)[s]$, if $j \in V_s$. And then initialize all the strategies with priority lower than $\alpha^\frown(j,f)$. Start $(j+1)$-cycle by choosing a fresh number $x_{\alpha,j+1}$ as the attacker and define $\sigma_s = \alpha^\frown(j+1,f)$.

Otherwise, define $\sigma_s(t+1)$ as $\alpha^\frown(j,f)$ and go to the next substage.

Case 2 If $\zeta = \beta$ is an $\mathcal{N}$-strategy, there are three subcases.

Subcase ($\beta 1$) Let (k,l) be the active cycle of lowest priority which is not initialized since the last β-stage. Now, implement the delayed permitting (if any) as follows:

(β2.1) If U has a change below the restraint of a cycle $(k',l') \leq_L (k,l)$, without loss of generality, assume (k',l') to be the highest priority one, then define $\sigma_s(t+1)$ as $\beta^\frown(k',l')$ and then go to the next substage, if cycle (k',l') has not received an l'-permission so far; and define $\sigma_s(t+1)$ as $\beta^\frown(k',l')$, if cycle (k',l') received an l'-permission before. In the second case, redefine $\psi_\beta(l')$ as $\psi_\beta(l'-1)[s]$ since l' is in V_s and define $\sigma_s = \beta^\frown(k',l')$, initialize all the nodes with lower priority than σ_s and go to the next stage.

(β2.2) If β2.1 fails and V has a change on a number $k' \leq k$ or $l' \leq l$ between the last β-stage and stage $s+1$, let i be the least one. Check the highest priority cycle (k',l') that get a V-permitting.

If $i = l'$ for (k',l'), then just remove the corresponding $\gamma_1(x)[s_0]$ from B_1 and x from X, put s_0 into A, where $s_0 < s$ is the stage at which x is enumerated into X. Now stop cycles $> (k',l')$, redefine $\Theta^U_{k'}(l')$ as 1 with same use, initialize all the nodes with lower priority than $\beta^\frown(k',l')$ and go to the next stage.

If $i = k'$ for (k',l'), then just remove s_0 from A and, put x into X and $\gamma_1(x)[s_0]-1$ into B_1, where $s_0 < s$ is the stage at which x is first enumerated into X. Now stop cycles $> (k',l')$, redefine $\Xi^U(k')$ as 1 with same use, initialize all the nodes with lower priority than $\beta^\frown(k',l')$ and go to the next stage.

Subcase ($\beta 2$) If Subcase $\beta 1$ fails, then check if it is a β-expansionary stage. If this is a β-expansionary stage, then check whether $\Delta_\beta^{A,U}$ is defined correctly.

If $\Delta_\beta^{A,U}$ is defined correctly, then define $\Delta_\beta^{A,U}(z)[s] = D_{j(\beta)}(z)[s]$ for all $z < l(\beta, s)$ with $\delta_\beta(z) = s$. Let (k, l) be the active cycle of lowest priority. Now let the outcome of β be (k, l) and go to the next substage.

If for some z, $\Delta_\beta^{A,U}(z) \neq D_{j(\beta)}(z)$, let z be the least such number. Let (k, l) be the active cycle of lowest priority. Now cycle (k, l) takes action as in $\mathcal{N}$-strategy:

First restrain $A \upharpoonright \delta_\beta(z)[s]$, $B_1 \upharpoonright \delta_\beta(z)[s]$ and $B_2 \upharpoonright \delta_\beta(z)[s]$ from other strategies. Then take actions depending upon the situations:

Case i If $D_{j(\beta)}(z)[s] = 1$, then set $\Theta_k^U(l) = V(l)[s]$ with $\theta_k(l)[s] = \delta_\beta(z)[s]$ ($\delta(z)_\beta[s] < s$). Let the outcome of β be $(k, l+1)$. Initialize all the nodes with lower priority than $\beta^\frown(k, l+1)$ and go to the next stage.

Case ii If $D_{j(\beta)}(z)[s] = 0$, then set $\Xi^U(k) = V(k)[s]$ with $\xi(k)[s] = \delta_\beta(z)[s]$. Let the outcome of β be $(k+1, 0)$. Initialize all the nodes with lower priority than $\beta^\frown(k+1, 0)$ and go to the next stage.

Subcase ($\beta 3$) If β is not an expansionary stage, then define $\sigma_s = \beta^\frown f$ and go to the next substage.

4.4 Verification

Let $TP = \liminf_s \sigma_{2s}$ be the true path of the construction. We first prove that TP is infinite and then verify that the construction given above satisfies all the requirements.

Lemma 4.1. *Let σ be any node on TP. Then*

(1) *σ can only be initialized finitely often.*

(2) *σ has an outcome $\mathcal{O}$ such that $\sigma^\frown\mathcal{O}$ is on TP.*

(3) *σ can initialize strategies below $\sigma^\frown\mathcal{O}$ at most finitely often.*

Proof. We prove this lemma by induction.

When $\sigma = \lambda$, the root of the priority tree, it is an $\mathcal{N}_0$-strategy. It is obvious that (1) is true. To show (2), for a contradiction, suppose that $\lambda^\frown\mathcal{O} \notin TP$ for any outcome of λ. This happens only when there are infinitely many λ-expansionary stages, λ runs infinitely many cycles and each of them runs finitely often. So either there is a k such that for every l there is a stage s_l no cycle (k, l) acts after stage s_l, or for every k there is a stage s_k such that no cycle (k, l) acts after stage s_k. In the former case, Θ_k^U computes V correctly. In the latter case, Ξ^U computes V correctly. Either of them implies that $V \leq_T U$, contradicts with $U <_T V$. Thus (2) is true for λ. If there are finitely many λ-expansionary stages, $\lambda^\frown f$ will be on TP. So (3) is clearly true. Otherwise, there are infinitely many λ-expansionary stages, by (2), $\lambda^\frown(k, l)$ is on TP for some $k, l \in \omega$. So there is a stage s after which no nodes to the left of (k, l) is visited again. Thus all nodes below $\lambda^\frown(k, l)$ can never be initialized after stage s. Therefore (3) is true for λ.

Now, for any $\lambda \subset \sigma \in TP$, let σ^- be the immediate predecessor of σ. By the induction hypothesis, there is a stage s_0 after which σ^- cannot be initialized. There are two cases.

Case 1. $\sigma^- = \beta$ is an $\mathcal{N}$-strategy.

Then only the strategies to the left of σ can initialize σ after stage s_0. Since $\sigma \in TP$, by the induction hypothesis, there are only finitely many strategies lie to left of σ and each of them can be visited finitely many often. So there is a stage $s_{>}s_0$ such that all the strategies can never be visited after stage s_1. Thus after stage s_1, σ cannot be initialized again.

By our construction, σ is a $\mathcal{P}$-strategy. The $\mathcal{P}$-strategy runs ω many cycles of form $(j, \infty) <_L (j, f)$, $j \in \omega$. It suffices to show that there is a cycle $(j, -)$ for $- \in \{\infty, f\}$ such that σ runs it infinitely many often. If not, then for every cycle (j, f) there is a stage s_j after which σ never run any j cycle. Thus the definition of $\Psi^U(j)[s_j]$ computes $V(j)$ correctly. And then $\Psi^U = V$, i.e., $V \leq_T U$, a contradiction. So (2) is true.

Besides the strategies lying to the left of σ, σ itself can initialize the strategies below $\sigma^\frown\mathcal{O}$. Let s be the stage after which no strategies to the left of $\sigma^\frown\mathcal{O}$ can be visited. Now let $t > s$ be the stage at which the $\mathcal{P}$-strategy σ put a number into X. Then after stage t, no strategy can initialize any strategy below $\sigma^\frown\mathcal{O}$. So (3) holds.

Case 2. $\sigma^- = \alpha$ is a $\mathcal{P}$-strategy.

Apply a similar argument to Case 1 and λ, it is easy to see that this lemma is also true when $\sigma^- = \alpha$ is a $\mathcal{P}$-strategy.

□

Lemma 4.2. *Let σ be any node on TP. Then if σ is a $\mathcal{P}$ or $\mathcal{N}$-strategy then the corresponding $\mathcal{P}$ or $\mathcal{N}$-requirement is satisfied.*

Proof. We prove this by induction. Let s be the least stage after which $\sigma^\frown\mathcal{O} \in TP$ is never initialized and no strategy to the left of $\sigma^\frown\mathcal{O}$ is visited again.

In case that σ is a $\mathcal{P}$-strategy, let $\mathcal{O} = (j, -)$ where $- \in \{\infty, f\}$ by the previous lemma. Let $s_1 \geq s$ be the stage at which $x = x_{\sigma,j}$ is defined. Then this x cannot be cancelled later.

If there is a stage $s_2 \geq s_1$ such that $V_{s_2}(j) = 0$ and after s_2, $\Phi^{A,U}_{n(\sigma)}(x)$ does not converge to 0, then the j-cycle can never take action after stage s_2. In this case, σ has outcome (j, f) on the true path TP. So $X(x) = 0$ and $\Phi^{A,U}_{n(\sigma)}(x)$ does not converge to 0. That is, $X \neq \Phi^{A,U}_{n(\sigma)}$. Thus $\mathcal{P}_{n(\sigma)}$-requirement is satisfied in this case.

On the other hand, whenever $\Phi^{A,U}_{n(\sigma)}(x)$ converges to 0 at any σ-stage, j-cycle will set a restraint to protect the associated computation and start $(j + 1)$-cycle simultaneously. Then wait for j to enter V. Since σ has outcome $(j, -)$ on TP, there is a stage at which either U changes below the corresponding use or j enters V. If $V(j) = 0$ then it must be true that U changes. In this case, the j-cycle goes from step 5 to 2 infinitely often. So $\Phi^{A,U}_{n(\sigma)}(x) \uparrow$, and hence $\mathcal{P}$ is satisfied via witness x. Now we assume that $V(j) = 1$. If j enters V first, then at the next σ-stage, x is enumerated into X. And then U will not have a change below that use mentioned previously. Otherwise, $(j, -)$, now is (j, f), could not be the final outcome of σ on TP. In this case, $X(x) = 1$ and $\Phi^{A,U}_{n(\sigma)}(x) = 0$. So $X \neq \Phi^{A,U}_{n(\sigma)}$, and hence $\mathcal{P}_{n(\sigma)}$-requirement is satisfied.

In case that σ is an $\mathcal{N}$-strategy, if there are only finitely many σ-expansionary stages, then the $\mathcal{N}_{n(\sigma)}$-requirement is satisfied obviously.

If $\Phi_e^{B_1,A,U} = \Phi_e^{B_2,A,U} = D_j$ where $n(\sigma) = \langle e, j\rangle$, let $\tau = \sigma^\frown\mathcal{O} \in TP$, we show that $\Delta_{\tau,j}^{A,U}$ is totally well-defined and computes D_j correctly. First note that there will be infinitely many expansionary stages. In this case, there is cycle (k,l) that is the final outcome of σ by using a similar argument in lemma 4.1. Then by the choice of s, $V_s \upharpoonright k = V \upharpoonright k$. Now, it suffices to show that for every x, $\Delta_{\tau,j}^{A,U}(x)$ can be defined at most finitely often.

For a contradiction, fix the least z such that $\Delta_{\tau,j}^{A,U}(z)$ is (re)defined infinitely often. This can happen only when U changes below (k,l)-restraint infinitely often. Because the final outcome of σ is (k,l), this z must be associated with the cycle (k,l). Let $s_3 \geq s$ be the least stage at which σ runs (k,l)-cycle with z as its attacker.

First, consider the case that $l \notin V$. Assume that $D_{j,s_3}(z) = 1$ at stage s_3. Then it must be the case that $\Delta_{\tau,j}^{A,U}(z)[s_3] = 0$ and $l(\sigma, s_3) > z$. Since $\sigma^\frown(k,l) \in TP$, there must be a stage $s_4 > s_3$ at which U changes below $\theta(z)[s_3] = \delta_\sigma(z)[s_3]$. (*Otherwise, σ will start a new cycle $(k, l+1)$. This cycle will be active forever. So the final outcome of σ lies to the right of (k,l), a contradiction.*) Let $s_5 > s_4$ be the next σ-expansionary stage. Then $\Delta_{\tau,j}^{A,U}(z)$ will be defined correct at stage s_5. Now, if $z \in D_j$, then later we are never to correct $\Delta_{\tau,j}^{A,U}(z)$. So $\Delta_{\tau,j}^{A,U}(z)$ is defined finitely often. Since D_j is *d.c.e.*, this z may exit D_j after s_5. If so, by a similar argument, we could show that $\Delta_{\tau,j}^{A,U}(z)[s_5]$ can be undefined, at some stage $s_6 > s_5$ say, via a U-change. Then σ can correct $\Delta_{\tau,j}^{A,U}(z)$ at the next σ-expansionary stage s_7. Now the definition of $\Delta_{\tau,j}^{A,U}(z)$ at stage s_7 must be the final one since D_j is *d.c.e.* set. Therefore, $\Delta_{\tau,j}^{A,U}(z)$ is defined finitely often.

Second, consider the case that $l \in V$. By the previous case, it suffices to consider that there is a stage $s' > s$ at which σ is visited and $V(l)$ changes before U-change . Then we will implement the (delayed) permitting and stop all cycles $(k', l') > (k,l)$. Assume $z \in D_{\sigma,j}$. If U does not change below the restraint of (k,l)-cycle later, then the final outcome of σ will be f since B_1, A respecting the restraint of (k,l)-cycle preserves the computation $\Phi^{B_1,A,U}(z) = 0$, a contradiction. So U must change below the restraint of (k,l)-cycle at stage, $s'' > s'$ say. Let the next σ-expansionary stage be $s''' > s''$. Then the definition of $\Delta^{A,U}_{\tau,j}(z)[s''']$ is the final version. So $\Delta^{A,U}_{\tau,j}(z)$ is undefined at most finitely often. Suppose $z \notin D_j$. Then $\Delta^{A,U}_{\tau,j}(z)$ will be undefined at most one more time than that in the case $z \in D_j$ (A similar argument to the previous paragraph). Therefore $\Delta^{A,U}_{\tau,j}(z)$ can be undefined at most finitely often in any case.

□

Lemma 4.3. *$A, B_1, B_2 \leq_T V \oplus U$.*

Proof. By our construction, for a given strategy σ, we implement the V-permitting only when σ is visited again even though this permission happens before σ is visited. So the proof will get involved in a delayed permission argument.

To show that $A \leq_T V \oplus U$, fix a number $n \in \omega$. It is enough to show that we can $V \oplus U$-computably determine whether $n \in A$ or not. In the course of the construction, only $\mathcal{N}$-strategies can put or extract numbers from A. Furthermore if n can be enumerated into A, then it must be that some $\mathcal{P}$-strategy, α say, enumerates a number x into X, and $\gamma_i(x)[n-1]$ into $B_{i,n}$ at stage n, and later this x is extracted from X by some $\mathcal{N}$-strategy $\beta < \alpha$ with $\beta < \alpha$ when n is enumerated into A. So if at stage n, no number is enumerated into X, then $n \notin A$. Otherwise, assume x is enumerated into X at stage n by some α, and $\gamma_i(x)[n-1]$ into $B_{i,n}$ simultaneously. Find out the highest priority $\mathcal{N}$-strategy $\beta \subset \alpha$ (if any) with $\beta^\frown\mathcal{O} \subseteq \alpha$ such that for some z with $\Delta^{A,U}_{\beta^\frown\mathcal{O}}(z)$ defined, the computations $\Phi^{B_1,A,U}_{e(\beta)}(z)$ and $\Phi^{B_2,A,U}_{e(\beta)}(z)$ are injured by the enumeration of $\gamma_i(x)$-uses. If this β does not exist, then n will not be in A.

Now we assume the existence of such $\mathcal{N}$-strategy $\beta \subset \alpha$. For the convenience in the following proof, we introduce the following notions and denotations. Given $\sigma \in T$, the *permission-bound at stage s*, $b_s(\sigma)$, is defined as

$$\max(\{k+1, l+1, j+1 : \exists \tau \leq_L \sigma \text{ (active } \tau^\frown(k,l) \leq_L \sigma$$

$$\text{or active } \tau^\frown(j,-) \leq_L \sigma, - \in \{\infty, f\})\}).$$

where "active" means that it is not initialized, cancelled or stopped at stage s. Let $S(\sigma_s)$ be the least stage such that

$$V_{S(\sigma_s)} \upharpoonright b_s(\sigma) = V \upharpoonright b_s(\sigma).$$

Then $S(\sigma_s)$ is V-computable.

To $U \oplus V$-computably find the stage s such that $n \in A \iff n \in A_s$, we need the following crucial claim:

Claim 4.4. *Assume that a $\mathcal{P}$-strategy σ runs cycle j at stage s_0 and $s_1 \geq \max\{s_0, S(\sigma_s), j+1\}$ is a U-true stage. If σ does not run cycle j at stage s_1, then after stage s_1, if σ runs cycle j again, at stage s_2 say, cycle j must have been cancelled or initialized between stage s_0 and s_2.*

A similar result is true for an $\mathcal{N}$-strategy σ instead of j cycle with (k,l) cycle.

This claim can be proved by induction on the length of σ. We leave it as an exercise (a similar proof refers to Lemma 5.2 in [9]).

Now we go back to the proof of $A \leq_T V \oplus U$. If the outcome $\mathcal{O} = f$ such that $\beta^\frown f \subseteq \alpha$, then it is obvious that $n \notin A$. Suppose that $\sigma = \beta^\frown(k,l) \subseteq \alpha$ for some $k,l \in \omega$. Now if $l \in V_n$ or $l \notin V$, then $n \notin A$. Assume l enters V at stage $t > n$, let $t' > t$ be the least U-true stage greater than both t and $S(\sigma_t)$. t' is $U \oplus V$-computable, and so it is V-computable. At stage t', if σ is not visited, or if σ is visited but does not run cycle (k,l), then by the claim, $n \in A$ iff $n \in A_{t'}$. If σ runs cycle (k,l) at stage t', it is easy to see that $n \in A$ is possible only when $n \in A_{t'}$, and so after stage t', n cannot get V-permission to enter A. Now it suffices to show that after stage t', n cannot be removed from A if $n \in A_{t'}$. By the choice of $t' > S(\sigma_t)$, there is no such V-permission to allow n exiting A. Therefore $A \leq_T U \oplus V$.

Since the arguments for $B_1 \leq_T V \oplus U$ and $B_2 \leq_T V \oplus U$ are similar, without loss of generality, we only show $B_1 \leq_T V \oplus U$ (actually, this case is more complicated since B_2 is in fact an *c.e.* set.). Fix a number $n \in \omega$, we just show how to $V \oplus U$-computably determine whether $n \in B_1$ or not. First note that if n is not a $\gamma_1(x)[s]$ or $\gamma_1(x)[s] - 1$ for some x, s, then $n \notin B_1$. Find the first x, s such that $\gamma_1(x)[s] > n$. If so far n or $n+1$ is not assigned as a use for some number $y < x$ at some stage $t < s$, then $n \notin B_1$.

Assume $\gamma_1(x)[s] = n$ for some x, s. By the construction, $\gamma_1(x)[s]$ is enumerated into B_1 only when x is enumerated into X at a stage $t(> s)$ before which $\gamma_1(x)[s]$ is not undefined. Now we find the least stage $s' \geq s$ at which some $\mathcal{P}$-strategy choose an attacker $y > x$. So if x is not chosen as an attacker till stage t, then $n \notin B_1$. Otherwise, let α be the $\mathcal{P}$-strategy that chooses $x = x_{\alpha,j}$ as an attacker for some cycle j at some stage s_0. If $j \in V_{s_0}$ or $j \notin V$, then it is clear $n \notin B_1$ since n cannot get V-permission.

Suppose that j enters V at stage $s_1 > s_0$, let $s_2 > s_1$ be the least U-true stage greater than both s_1 and $S(\alpha_{s_1})$. s_2 is $U \oplus V$-computable. At stage s_2, if α is not visited, or if α is visited but does not run cycle j, then by the claim, $n \in B_1$ iff $n \in B_{1,s_2}$. If α runs cycle j at stage s_2, it is easy to see that $n \in B_1$ is possible only when $n \in B_{1,s_2}$, and so after stage s_2, n cannot get V-permission to enter B_1. Thus if $n \notin B_{1,s_2}$ then it must be $n \notin B_1$.

Now assume $n \in B_{1,s_2}$. If the enumerations of $\gamma_i(x)$ into B_i could not injure any computations $\Phi^{B_1,A,U}_{e(\beta)}(z)$ and $\Phi^{B_2,A,U}_{e(\beta)}(z)$ simultaneously for some z such that $\Delta^{A,U}_{\beta^\frown\mathcal{O}}(z)$ defined where $\beta \subset \alpha$ is an $\mathcal{N}$-strategy and $\beta^\frown\mathcal{O}$ is active, then $n \in B_1$. Otherwise, let β be such an $\mathcal{N}$-strategy. Without loss of generality, suppose that $\mathcal{O} = (k,l)$ for some $k,l \in \omega$. Let $\sigma = \beta^\frown(k,l)$. Now let $s_3 > s_2$ be the least U-true stage greater than both s_2 and $S(\sigma_{s_2})$. s_3 is $U \oplus V$-computable. At stage s_3, if β is not visited, or if β is visited but does not run cycle (k,l), then by the claim, $n \in B_1$ iff $n \in B_{1,s_3}$. If α runs cycle (k,l) at stage s_3, it is easy to see that $n \in B_1$ iff $n \in B_{1,s_3}$, and so after stage s_3, n cannot get V-permission to exit B_1.

Assume $\gamma_1(x)[s] = n+1$ for some x, s. We now first check everything for $\gamma_1(x)[s]$ as above till an alike stage s_3. Now if $\gamma_1(x)[s] \in B_{1,s_3}$, then $n = \gamma_1(x)[s] - 1$ must not be in B_1. Otherwise, find the number s'' that is enumerated into A when $\gamma_1(x)[s]$ is removed from B_1. Now ask V whether s'' is in A or not. By $A \leq_T V$, it can be V-computably answered. Then $n \in B_1$ iff $s'' \notin A$. Therefore $B_1 \leq_T U \oplus V$.

□

Lemma 4.5. *$\Gamma^{B_1,A,U}_1$ and $\Gamma^{B_2,A,U}_2$ are well-defined. Furthermore, they are defined correctly, that is, $X = \Gamma^{B_1,A,U}_1 = \Gamma^{B_2,A,U}_1$. So $X \leq_T V$.*

Proof. As the arguments are similar for showing $\Gamma^{B_1,A,U}_1$ and $\Gamma^{B_2,A,U}_2$ are well-defined, without loss of generality, we only show that $\Gamma^{B_1,A,U}_1(x)$ is undefined at most finitely often for all $x \in \omega$. We prove this lemma by induction. Fix a number $x \in \omega$.

Suppose for all $y < x$, $\Gamma_1^{B_1,A,U}(y)$ can be undefined at most finitely often. Let s be the least stage after which each $\Gamma_1^{B_1,A,U}(y)$ for $y < x$ cannot be undefined again. Assume that $\Gamma_1^{B_1,A,U}(x)$ is undefined by some $\mathcal{P}$-strategy, α say, enumerating x into X at a stage $t > s$. (*If this does not happen, then after stage* s, $\Gamma_1^{B_1,A,U}(x)$ *will never be undefined once it is defined.*) Then the new definition of $\Gamma_1^{B_1,A,U}(x)$ after stage t can be undefined only when x exits X since then. Now, $\Gamma_1^{B_1,A,U}(x)$ can be undefined at most once, that is, later x is enumerated X again. So $\Gamma^{B_i,A,U}$ are totally defined.

Given $x \in \omega$. If a $\mathcal{P}$-strategy put x into X at stage s, then we will undefine $\Gamma_i^{B_i,A,U}(x)[s-1]$ by enumerating its use into B_i. So $\Gamma_i^{B_i,A,U}(x)$ will be defined correctly later in this case. When an $\mathcal{N}$-strategy, β say, extracts x from X at a stage $t > s$, it will remove $\gamma_1(x)[s-1]$ from B_1 or put s into A to undefine the new definitions of $\Gamma_i^{B_i,A,U}(x)$, and redefine them same as $X_t(x)$. If β enumerates x into X again at a stage $t' > t$, β also puts $\gamma_1(x)[s-1]-1$ into B_1 and extracts s from A. Now $\Gamma_1^{B_1,A,U}(x)$ is redefined as same as $\Gamma_1^{B_1,A,U}(x)[s-1] = 1 = X(x)$, $\Gamma_2^{B_2,A,U}(x)$ is also redefined same as $X(x)$. Therefore, each $\Gamma_i^{B_i,A,U}$ is defined correctly at each stage.

□

Lemma 4.6. *$X \not\leq_T A \oplus U$ and $A \oplus U \equiv_T B_1 \oplus A \oplus U \wedge B_2 \oplus A \oplus U$ in $\mathbf{D}_2$.*

Proof. From the $\mathcal{P}$-strategies, $X \not\leq_T A \oplus U$ holds. The $\mathcal{N}$-strategies guarantee that $A \oplus U \equiv_T B_1 \oplus A \oplus U \wedge B_2 \oplus A \oplus U$ in $\mathbf{D}_2$ as $\{D_j\}_{j \in \omega}$ list all *d.c.e.* sets.

□

This completes the proof of Theorem 4. ◇◇

Chapter 5

AN ALMOST UNIVERSAL CUPPING DEGREE

5.1 Introduction

In this chapter, we investigate the cupping property of the d.c.e. degrees. In [52], Li, Song and Wu showed the existence of incomplete ω-c.e. degrees cupping every incomputable c.e. degree to $\mathbf{0}'$, where they called these degrees universal cupping degrees. Obviously, each universal cupping degree is a Yates degree, where a Δ^0_2 degree is called a Yates degree if $\mathbf{0}$ and $\mathbf{0}'$ are the only two c.e. degrees comparable to it. Slaman and Steel proved in [82] that Yates degrees can be 1-generic. In [90], Wu proved that Yates degrees can occur in every jump class. Yates degrees cannot fall in any n-c.e. level in the Ershov hierarchy, by the existence of Lachlan sets. A variant of Yates degree at the n-c.e. level, called bi-isolated degrees, was introduced by Wu in [88]. An n-c.e. degree $\mathbf{d}$ is called *bi-isolated* if there is a c.e. degree $\mathbf{a}$ such that there are c.e. degrees neither between $\mathbf{a}$ and $\mathbf{d}$, nor between $\mathbf{d}$ and $\mathbf{0}'$.

Obviously, universal cupping degrees cannot be any n-c.e. degree, by Lachlan's observation that each nonzero n-c.e. degree bounds a nonzero c.e. degree. Here, we show the existence of almost universal cupping d.c.e. degrees, and that such degrees can even be isolated.

Theorem 5 (Liu and Wu [57]) There is an incomplete d.c.e. set D and a c.e. set B such that $D \not\leq_T B$, and for any c.e. set W_e, either $B \oplus D \oplus W_e \equiv_T \emptyset'$ or $W_e \leq_T B$.

Obviously, each isolated almost universal cupping d.c.e. degree is bi-isolated. We comment here that any incomplete maximal d.c.e. degree, as constructed in Cooper et al. in [16] is an almost universal cupping degree, but its converse is not true by Theorem 1.21 and Theorem 5. We do not know whether the construction given in [16] can be modified to make that all the c.e. degrees below an incomplete maximal d.c.e. degree bounded by a c.e. degree.

5.2 Requirements

To prove Theorem 5, we will construct a d.c.e. set D, a c.e. set B and an auxiliary set A, partial computable functionals $\{\Gamma_e\}_{e\in\omega}$ and $\{\Delta_e\}_{e\in\omega}$ satisfying the following requirements:

$\mathcal{P}_e$: $A \neq \Phi_e^B$,

$\mathcal{I}_e$: $W_e = \Phi_e^{B,D} \Rightarrow \exists\Theta_e(W_e = \Theta_e^B)$,

$\mathcal{R}_e$: $K = \Gamma_e^{B,D,W_e} \vee W_e = \Delta_e^B$.

where K is a creative set and $\{W_e\}_{e\in\omega}$ is an effective list of all c.e. sets. The $\mathcal{P}_e$-requirements ensure that B is incomplete. The $\mathcal{R}$-requirement ensures that every c.e. set W_e either cups $B \oplus D$ to K via Γ_e (constructed by us) or is Turing reducible to B witnessed by Δ_e (constructed by us). The $\mathcal{R}_e$-requirements ensure that $B \oplus D$ is a proper d.c.e. set, and that no incomplete c.e. degree is above the degree of $B \oplus D$. The $\mathcal{I}_e$-requirements ensure that $B \oplus D$ is isolated by B. That is, no c.e. degree is between degrees of B and $B \oplus D$.

5.3 Strategies

In this section, we describe how each strategy works and consider the interactions between various strategies.

5.3.1 A $\mathcal{P}$-strategy

A $\mathcal{P}$-strategy is simply a Friedberg-Muchnik's diagonalization argument. Let α be a $\mathcal{P}_e$-strategy - for convenience, we use $\Phi_\alpha^B(x)$ to denote $\Phi_{e(\alpha)}^B(x)$ - this convention also applies to other strategies.

α works as follows:

1. Choose a witness x, and wait for $\Phi_\alpha^B(x)\downarrow= 0$.
2. Put x into A and protect B.

There are two possible cases.

1. α waits forever at step 1. Then $\Phi_\alpha^B(x)$ never converges to 0, and hence $A(x) = 0 \neq \Phi_\alpha^B(x)$, $\mathcal{P}_e$ is satisfied. Let w denote this outcome.
2. α reaches step 2 at some stage. Then $\Phi_\alpha^B(x)\downarrow= 0 \neq 1 = A(x)$, and again, $\mathcal{P}_e$ is satisfied. Let d denote this outcome.

5.3.2 An $\mathcal{R}$-strategy

Let β be an $\mathcal{R}_e$-strategy. For the given W_e, β will construct a partial computable functional Γ_β (we use Γ_β to denote $\Gamma_{e(\beta)}$ and we use W_β to denote $W_{e(\beta)}$for simplicity) such that $K = \Gamma_\beta^{B,D,W_\beta}$. Γ_β is constructed as follows:

1. At stage s, define $\Gamma_\beta^{B,D,W_\beta}(z)[s] = K_s(z)$ for all $z < s$ such that $\Gamma_\beta^{B,D,W_\beta}(z)[s]\uparrow$, with use $\gamma_\beta(B\oplus D\oplus W_\beta; z)[s]$ ($\gamma_\beta(z)[s]$ for short) as a fresh number.
2. If $\Gamma_\beta^{B,D,W_\beta}(z)[s]\downarrow\neq K_s(z)$, then put $\gamma_\beta(z)[s]$ into D to undefine $\Gamma_\beta^{B,D,W_\beta}(z)$.

In the construction, β may enumerate uses $\gamma_\beta(z)$ into D to ensure that $\Gamma_\beta^{B,D,W_e}(z)$ equals to $K(z)$ for infinitely many $z \in \omega$, if $\Gamma_\beta^{B,D,W_\beta}(z)$ is defined. This will cause an obvious conflict between β and those $\mathcal{I}$-strategies η below β, since η attempts to restrain D from changes to preserve a computation. This kind of interactions is the major concern of the whole construction, as we will see soon after introducing the $\mathcal{I}$-strategies.

5.3.3 An $\mathcal{I}$-strategy

Let η be an $\mathcal{I}_e$-strategy for some $e \in \omega$. We write W_η for $W_{e(\eta)}$ and $\Phi_\eta^{B,D}$ for $\Phi_{e(\eta)}^{B,D}$ for convenience. Given a stage s, define the length of agreement function $l(\eta, s)$ over η as

$$l(\eta, s) = \max\{x \mid \forall y < x(W_{\eta,s}(y) = \Phi_\eta^{B,D}(y)[s] \downarrow)\},$$

and the maximal length function

$$m(\eta, s) = \max\{l(e, t), 0 \mid t < s \text{ and } t \text{ is an } \eta\text{-stage}\},$$

where t is called an *η-stage* if η is visited at stage t. We say s is an *η-expansionary stage* if s is an η-stage and $l(\eta, s) > m(\eta, s)$. For convenience, set $\varphi_{\eta,s}(B, D; l(\eta, s)) = \max\{\varphi_\eta(B, D; x)[s] \mid x < l(\eta, s)\}$.

The basic strategy for η is to extend the definition of Θ_η at each η-expansionary stage s such that $\Theta_\eta^B \upharpoonright l(\eta, s) = W_{\eta,s} \upharpoonright l(\eta, s)$. However, in the construction, after stage s, numbers enumerated into D or B may injure the current convergent computation $\Phi_\eta^{B,D} \upharpoonright l(\eta, s)[s]$. In case that a smaller number is enumerated into B, then Θ_η^B will be undefined correspondingly. A problem arises when numbers, z say, are enumerated into or removed from D, as some $\Phi_\eta^{B,D}(y)$, $y < l(\eta, s)$, is injured, leading to a change of both $\Phi_\eta^{B,D}(y)$ and $W_\eta(y)$, from 0 to 1. In this case, as no number is enumerated into B, $\Theta_\eta^B(y)$ is kept the same, and equals to 0, which shows that Θ_η^B does not compute W_η correctly.

We get around this problem as follows. Assume that $\Theta_\eta^B(y)[s] = 0$ is defined at η-expansionary stage s. That is, $\Phi_\eta^{B,D}(y) = W_{\eta,s}(y) = 0$. Suppose that $s_1 > s$ is the least η-expansionary stage at which $W_{\eta,s_1}(y) = 1$. This means that between stages s and s_1, there is a stage s_0 at which a number less than $\varphi_\eta(y)[s]$ is enumerated into D. In this case, η removes all numbers enumerated into D after stage s_0 to recover the computation $\Phi_\eta^{B,D}(y)$ to $\Phi_\eta^{B,D}(y)[s]$, which produces an inequality between $W_\eta(y)$ and $\Phi_\eta^{B,D}(y)$ as

$$\Phi_\eta^{B,D}(y)[s_1] = \Phi_\eta^{B,D}(y)[s] = 0 \neq 1 = W_{\eta,s_1}(y) = W_\eta(y).$$

Such a y (the least) is called *a witness or a potential disagreement point* for the isolation strategy η. To preserve this inequality, it is sufficient to protect the computation $\Phi_\eta^{B,D}(y)$ as $\Phi_\eta^{B,D}(y)[s_1]$ since W_η is a c.e. set, and $W_\eta(y)$ cannot be changed back to 0.

η has three outcomes f, d and ∞, where f means that there are only finitely many η-expansionary stages, ∞ means that there are infinitely many η-expansionary stages and d means that η succeeds in creating a disagreement between $\Phi_\eta^{B,D}$ and W_η as above. In this simple case, these three outcomes have the priority:

$$\infty <_L f <_L d.$$

5.3.4 The interaction among $\mathcal{I}$-strategies

The crucial point of the construction is that whenever numbers are extracted from D, some small numbers will be enumerated into B, to ensure the consistency between different strategies. Here we consider the necessity of enumerating numbers into B for the sake of the consistency between $\mathcal{I}$-strategies, and we will further discussion on this when $\mathcal{R}$-strategies are considered.

Recall that when $\Theta_\eta^B(y)$ is defined at stage s, we have $\Phi_\eta^{B,D}(y)[s] = W_{\eta,s}(y)$ and we define $\Theta_\eta^B(y)[s] = W_{\eta,s}(y)$. This computation $\Phi_\eta^{B,D}(y)$ can later be injured by the changes of D or B, at a bigger stage $s' > s$. An enumeration into B first will not cause trouble as this enumeration will also undefine $\Theta_\eta^B(y)$, which allows us to redefine $\Theta_\eta^B(y)$ again at a later stage to make it equal to $\Phi_\eta^{B,D}(y)$. We have seen that when the computation $\Phi_\eta^{B,D}(y)$ is injured because of an enumeration of a small number into D, as described above, we can extract such a number from D to recover D, and hence to recover a computation to $\Phi_\eta^{B,D}(y)[s]$. However, such an extraction action satisfies η by creating a disagreement, but on the other hand, it also causes problems to those $\mathcal{I}$-strategies with higher priority. To see this, let $\eta' < \eta$ be an $\mathcal{I}$-strategy with $\eta'^\frown\infty \subset \eta$, so η acts only at η'-expansionary stages.

1. At an η-expansionary stage s_0 (of course it's also η'-expansionary), a number z is enumerated into D.

2. At an η'-expansionary stage $s_1 > s_0$, η' defines $\Theta^B_{\eta'}(y')[s_1] = W_{\eta',s_1}(y')$, as 0 (for a general purpose), with use $\theta_{\eta'}(B; y')[s_1] = s_1$ for some y'. We assume that the computation $\Phi^{B,D}_{\eta'}(y')[s_1]$ uses the fact that z is in D_{s_1}.

3. At an η-expansionary stage $s_2 > s_1$, η sees that $W_{\eta,s_2}(y) \neq \Theta^B_\eta(y)[s_2]$, and hence η recovers D to D_{s_0} *to recover the* $\Phi^{B,D}_\eta(y)$ *to* $\Phi^{B,D}_\eta(y)[s_0]$ *to create a disagreement between* $\Phi^{B,D}_\eta(y)$ *and* $W_\eta(y)$*.* In particular, z is extracted from D.

4. At an η'-expansionary stage $s_3 > s_2$, η' sees that $W_{\eta',s_3}(y') \neq \Theta^B_{\eta'}(y')[s_3]$. This is so because z is removed from D and $\Phi^{B,D}_{\eta'}(y')[s_3]$ is a new computation, converging to 1.

As we are constructing D as a d.c.e. set, z cannot be enumerated into D again to recover $\Phi^{B,D}_{\eta'}(y')$ to $\Phi^{B,D}_{\eta'}(y')[s_2]$. To avoid such a situation, at stage s_2, when z is extracted from D, we also enumerate s_0, the stage at which z is enumerated into D, into B. This enumeration of s_0 into B undefines $\Theta^B_{\eta'}(y')$ as s_0 is less than s_1, and hence the inequality in step 4 can never occur. This ensures that η's action is consistent with those $\mathcal{I}$-strategies with higher priority. Also note that extra enumeration of s_0 into B is also consistent with η itself, as s_0 is bigger than the use of $\Phi^{B,D}_\eta(y)[s_0]$.

Another point is that there are perhaps infinitely many $\mathcal{I}$-strategies below the outcome ∞ of η, and we need to make sure that for a given $y \in \omega$, $\Theta^B_\eta(y)$ can be undefined in this way by only finitely many such $\mathcal{I}$-strategies. Note that after η sees that $\Phi^{B,D}_\eta(y)$ converges, and when $\Theta^B_\eta(y)$ is first defined at stage s, only finitely many $\mathcal{I}$-strategies are visited below η's outcome ∞, and only these strategies can undefine $\Theta^B_\eta(y)$ later, as the use $\theta_\eta(B; y)$ is always defined as s, and hence after all these $\mathcal{I}$-strategies stop to act, $\Theta^B_\eta(y)$ cannot be undefined anymore.

5.3.5 One $\mathcal{I}$-strategy below one $\mathcal{R}$-strategy

Now we consider how an $\mathcal{I}$-strategy η works below an $\mathcal{R}$-strategy β. Let s be an η-expansionary stage at which $\Theta^B_\eta(y)[s] = 0$ is defined. Assume that $s_0 > s$ is a stage at which β would put some small number $\gamma_\beta(k)[t] < \varphi_\eta(y)[s]$ into D as k enters K

currently, which may change $\Phi_\eta^{B,D}(y)$ and injures η's strategy. Such a process can happen infinitely many times, and hence η cannot satisfy the requirement $\mathcal{I}_e$. To avoid this, we need to make the computation $\Phi_\eta^{B,D}(y)$ clear of the γ_β-uses, and we apply the threshold strategy introduced as follows.

When η is first visited, it fixes a parameter p_η as a threshold for the enumeration of γ_β-uses. Whenever a number $k \leq p_\eta$ enters K, we enumerate the current $\gamma_\beta(p_\eta)$-use into D and *reset the strategy* η by cancelling all the parameters defined at η, except for p_η, and initializing all strategies with lower priority. Once the parameter p_η is settled down, this resetting process can happen at most finitely often. Suppose that after stage s', η cannot be reset. Assume that s and s_0 are as in last paragraph with $s, s_0 > s'$. Then at the least η-expansionary stage $s_1 \geq s_0$, η puts $\gamma_\beta(p_\eta)[s_1]$ into D to *start an attack* on β with outcome g_β by defining

$$\Delta_{\eta\beta}^B \restriction \gamma_\beta(p_\eta)[s_1] = W_{\beta,s_1} \restriction \gamma_\beta(p_\eta)[s_1]$$

with use s_1 and waiting for a W_β-change below $\gamma_\beta(p_\eta)[s_1]$. We say that the attack associated with $\gamma_\beta(p_\eta)[s_1]$ is *activated* at stage s_1.

There are two possible cases.

1. If $W_\beta \restriction \gamma_\beta(p_\eta)[s_1]$ has not changed since stage s_1, then the attack associated with $\gamma_\beta(p_\eta)[s_1]$ remains active and in this case, η fails to pass the threshold p_η for β but succeeds in defining $\Delta_{\eta\beta}^B$ up to $\gamma_\beta(p_\eta)[s_1]$.

2. Otherwise, there is an η-expansionary stage $t > s_1$,

$$W_{\beta,s_1} \restriction \gamma_\beta(p_\eta)[s_1] \neq W_{\beta,t} \restriction \gamma_\beta(p_\eta)[s_1].$$

   Then η uses this W_β-change, *instead of enumerating* $\gamma_\beta(p_\eta)[s_1]$ *into* D, to lift the value of $\gamma_\beta(p_\eta)$. η's action is to remove all numbers being enumerated into D after stage s to recover the computation $\Phi_\eta^{B,D}(y)$ to $\Phi_\eta^{B,D}(y)[s]$, where, as indicated above, s is the stage indicated in the last section. Then after stage t, the enumeration of γ_β-uses will not affect the computation $\Phi_\eta^{B,D}(y)[s]$ (*by the* *choice of s', no $k \leq p_\eta$ can enter K after stage s').* We say that the attack is completed at stage t, and that η *passes threshold p_η for β.*

Accordingly, η has one more outcome:

g_β: η starts infinitely many attacks but fails to pass the threshold p_η for β. Then $\Delta^B_{\eta\beta}$ is totally defined and computes W_β correctly — $\mathcal{R}_e$ is satisfied, even though $\Gamma^{B,D,W_\beta}_\beta(p_\eta)$ diverges.

To create a disagreement between $\Phi^{B,D}_\eta$ and W_η at y (to have outcome d), we need to have the corresponding computation $\Phi^{B,D}_\eta(y)$ clear of all the γ_β-uses, and η's purpose is to make this computation pass the threshold p_η for β. η can now have four outcomes with the priority:

$$g_\beta <_L \infty <_L f <_L d.$$

If η has outcome g_β, then $\mathcal{I}_e$ is not satisfied at η, and a backup strategy for the requirement $\mathcal{I}_e$ is given below the outcome g_β. Let η' be such a backup strategy below $\eta^\frown g_\beta$. Then η' knows that that eventually, (almost) all γ_β-uses will go to infinity — in particular, $\gamma_\beta(p_\eta)$ goes to infinity. η' only believes a computation $\Phi^{B,D}_{\eta'}(y)$ at stage s if $\gamma_\beta(p_\eta)[s]$ is bigger than $\varphi_{\eta'}(y)[s]$ — we will call such a computation η'-believable in the remainder of this paper. By only using η'-believable computations, η' satisfies the requirement $\mathcal{I}_e$ in the usual way, as β's further enumeration will not affect such computations in the later construction.

5.3.6 An $\mathcal{I}$-strategy below many $\mathcal{R}$-strategies

We now consider the case when an $\mathcal{I}$-strategy η is below several $\mathcal{R}$-strategies $\beta_0 \subset \cdots \subset \beta_n$. Then, when η sees that y is a potential witness for the disagreement between $\Phi^{B,D}_\eta$ and W_η, η needs to ensure that the computation $\Phi^{B,D}_\eta(y)$ is clear of all the γ_{β_j}-uses for $0 \leq j \leq n$, so that after this disagreement is established, it will not be injured by these β-strategies. As before, η first sets a threshold p_η (*again, we reset η by cancelling all parameters and cycles of η, except for p_η, when a number $k \leq p_\eta$*

enters K), and to ensure that no such a β_j can enumerate a number into D, injuring computation $\Phi_\eta^{B,D}(y)$, we need to force a change of W_{β_j}, for each j, to have a small change to lift these γ_{β_j}-uses. η's actions consists of several, perhaps infinitely many, cycles, beginning with cycle 0. Each cycle i can start cycle $i+1$, and can also cancel all cycles $i' > i$ — when the latter happens, a (small) number will be enumerated into B, for the sake of consistency among strategies. The numbers enumerated into B are relatively large and will not affect η's purpose of creating an inequality. As we will see soon, this kind of enumeration into B is the key point ensuring that the strategies consistent.

Fix $i \geq 0$, and cycle i works as follows:

$i1$. At η-expansionary stages s, extend the definition of Θ_η^B up to $l(\eta, s)$ if $\Theta_\eta^B[s]$ and $W_{\eta,s}$ agree on all the numbers in the domain of $\Theta_\eta^B[s]$. In this case, β has outcome ∞.

$i2$. At an η-expansionary stage s_1, η sees a (least) y with $\Theta_\eta^B(y)$ defined but not equal to $W_{\eta,s_1}(y)$, then this y is a potential witness, selected by cycle i, for the disagreement between $\Phi_\eta^{B,D}$ and W_η. η starts to lift the γ_{β_j}-uses by running the following steps, from $j = n$. Each step can have two phases, attacking phase $i2-j1$ (at stage s_{n-j}^1) and completing phase $i2-j2$ (at stage s_{n-j}^2):

$i2-j1$. *Attack on* β_j.

Enumerate $\gamma_{\beta_j}(p_\eta)[s_{n-j}^1]$ into D (a capricious enumeration) and extend the definition of $\Delta_{\eta\beta_j}^B$ by letting

$$\Delta_{\eta\beta_j}^B(z)[s_{n-j}^1] = W_{\beta_j, s_{n-j}^1}(z)$$

with use s_{n-j}^1 for those $z < \gamma_{\beta_j}(p_\eta)[s_{n-j}^1]$ and $\Delta_{\eta\beta_j}^B(z)$ is not defined yet. β has outcome g_j. Wait for W_{β_j} to change below $\gamma_{\beta_j}(p_\eta)[s_{n-j}^1]$, and at the same time start cycle $i+1$.

$i2-j2$. *Completion on* β_j.

At η-expansionary stage $s^2_{n-j} > s^1_{n-j}$, η sees that

$$W_{\beta_j, s^1_{n-j}} \upharpoonright \gamma_{\beta_j}(p_\eta)[s^1_{n-j}] \neq W_{\beta_j, s^2_{n-j}} \upharpoonright \gamma_{\beta_j}(p_\eta)[s^1_{n-j}],$$

(*We will use this W_{β_j}-change to lift the use $\gamma_{\beta_j}(p_\eta)$*).
Extract from D all numbers, including $\gamma_{\beta_j}(p_\eta)[s^1_{n-j}]$, enumerated into D after stage s^1_{n-j} *to recover the computation $\Phi^{B,D}_\eta(y)$ to $\Phi^{B,D}_\eta(y)[s]$, which is now clear of γ_{β_j}-uses.* Also enumerate s^1_{n-j} into B for the sake of consistency between strategies. Cancel all cycles $i' > i$. We say that cycle i passes the threshold p_η for β_j.

If $j > 0$, then run step $j-1$. In this case, β has outcome g_{j-1}.

If $j = 0$, then extract from D the numbers enumerated into D between stages s (the last η-expansionary stage at which η sees that Θ^B_η is correct) and s^1_n (when we started attack on β_n). Enumerate s into B and declare that η is satisfied. From now on, η will always have outcome d.

Now η has the corresponding Σ_3 outcomes

$$g_0 <_L g_1 <_L \cdots <_L g_n,$$

all of which are to the left of $\infty <_L f <_L d$ (we write g_j for g_{β_j}).

If η runs only finitely many cycles, then either a cycle i passes the threshold p_η for all β_j, $0 \leq j \leq n$, in which case $i2-02$ is reached for cycle i and a disagreement is created, η is satisfied, or a cycle i never goes to phase $i2$, which means that there are only finitely many η-expansionary stages or Θ^B_η defined by this cycle i always agrees with W_η on all arguments, and we can argue as usual that η is also satisfied.

If η runs infinitely many cycles, then as each cycle has only finitely many possible outcomes, one of them (least), g_j say, will be the true outcome for infinitely many cycles, which means $\Delta^B_{\eta\beta_j}$ will be defined infinitely often. Let s' be the least stage after which no cycle can have outcome $g_{j'}$ with $j' < j$. It is easy to see that $\Delta^B_{\eta\beta_j}$ computes W_{β_j} correctly, by the choice of s' and also our assumption on j. In this case, η shows that W_{β_j} is reducible to B, and hence β_j is satisfied at η.

At the end of this section, we classify when a cycle is considered as active. Let $i_1 < i_2$ be two cycles and suppose that cycle i_1 is waiting for a W_{β_j}-change. Cycle i_1 becomes inactive automatically when cycle i_2 starts its attack on β_j. This means that when cycle i_1 becomes inactive, then we give up the potential disagreement of cycle i_1, which is reasonable in sense that we now turn to use the potential disagreement of cycle i_2. Note that if a cycle i_1 is waiting for a W_{β_k}-change, then the cycle i_1 still keeps active whenever a cycle $i_2 > i_1$ starts its attack on β_l for $l > k$.

5.3.7 Two $\mathcal{I}$-strategies below two $\mathcal{R}$-strategies

We now consider how two $\mathcal{I}$-strategies η_1, η_2 work below two $\mathcal{R}$-strategies β_1, β_2, or even more. A nontrivial case is when $\beta_1 \subset \beta_2 \subset \eta_1\widehat{\ }g_2 \subset \eta_2$. Let p_{η_1}, p_{η_2} be the thresholds on η_1 and η_2 respectively. Also let $y_{1,s}$ and $y_{2,s}$ be potential disagreements seen by cycles, i_{η_1}, i_{η_2}, of η_1 and η_2 at stage s, respectively.

The first problem is as follows.

1. At an η_2-expansionary stage s_1, η_k sees $\Theta^B_{\eta_k}(y_{k,s_1})[s_1] \neq W_{\eta_k,s_1}(y_{k,s_1})$ for $k = 1, 2$.

   At this stage, for $k = 1, 2$, both cycles i_{η_k} start their attacks — cycle i_{η_k} enumerates $\gamma_{\beta_{3-k}}(p_{\eta_k})[s_1]$ into D, and also extend the definition of $\Delta^B_{\eta_k\beta_{3-k}}$ for those $z < \gamma_{\beta_{3-k}}(p_{\eta_k})[s_1]$ with $\Delta^B_{\eta_k\beta_{3-k}}(z)$ is currently not defined, and wait for a change of $W_{\beta_{3-k}}$ below $\gamma_{\beta_{3-k}}(p_{\eta_k})[s_1]$.

2. At an η_1-expansionary stage $s_2 > s_1$, cycle i_{η_1} sees a change of W_{β_2}, i.e.

   $$W_{\beta_2,s_1} \upharpoonright \gamma_{\beta_2}(p_{\eta_1})[s_1] \neq W_{\beta_2,s_2} \upharpoonright \gamma_{\beta_2}(p_{\eta_1})[s_1].$$

   So $\Delta^B_{\eta_1}$ becomes incorrect, and η_1 wants to recover the computation $\Phi^{B,D}_{\eta_1}(y_{1,s_1})$ to the one at stage s_0 (where s_0 is the stage at which $\Theta^B_{\eta_1}(y_{1,s_1})[s_1]$ was defined). η_1's extraction of only $\gamma_{\beta_2}(p_{\eta_1})[s_1]$ from D is not enough for this purpose, as $\gamma_{\beta_1}(p_{\eta_2})[s_1]$ is still in D.

To avoid this, at stage s_2, η_1 also extracts $\gamma_{\beta_1}(p_{\eta_2})[s_1]$ out of D. η_1 can take such an extraction because η_2 is initialized at stage s_2.

The second problem occurs when more strategies are involved, which shows the necessity of enumerating numbers into B, when some numbers are removed from D. We already include such enumerations in the previous discussions.

1. At an η_2-expansionary stage s_0, η_2 sees that $\Theta^B_{\eta_2}(y_{2,s_0})[s_0] \neq W_{\eta_2,s_0}(y_{2,s_0})$. η_2 enumerates $\gamma_{\beta_1}(p_{\eta_2})[s_0]$ into D to start an attack for cycle i_{η_2}, extend the definition of $\Delta^B_{\eta_2\beta_1}$ and wait for a change of W_{β_1} below $\gamma_{\beta_1}(p_{\eta_2})[s_0]$.

2. At an η_1-expansionary stage $s_1 > s_0$, η_1 sees that $\Theta^B_{\eta_1}(y_{1,s_1})[s_1] \neq W_{\eta_1,s_1}(y_{1,s_1})$. η_1 enumerates $\gamma_{\beta_2}(p_{\eta_1})[s_1]$ into D to start an attack for cycle i_{η_1}, extend the definition of $\Delta^B_{\eta_1\beta_2}$ and wait for a change of W_{β_2} below $\gamma_{\beta_2}(p_{\eta_1})[s_1]$.

3. At an η_2-expansionary stage s_2, η_2 sees that

$$W_{\beta_1,s_0} \upharpoonright \gamma_{\beta_1}(p_{\eta_2})[s_0] \neq W_{\beta_1,s_2} \upharpoonright \gamma_{\beta_1}(p_{\eta_2})[s_0].$$

   Then η_2 extracts $\gamma_{\beta_1}(p_{\eta_2})[s_0]$ from D to recover the computation $\Phi^{B,D}_{\eta_2}(y_{2,s_0})$ to $\Phi^{B,D}_{\eta_2}(y_{2,s_0})[s_0]$, and hence complete the cycle i_{η_2}. *η_2 creates a disagreement between $\Phi^{B,D}_{\eta_2}$ and W_{η_2} and η_2 is satisfied. However, it may be true that $\varphi_{\eta_1}(B, D; y_{1,s_1})[s_1] > \gamma_{\beta_1}(p_{\eta_2})[s_0]$, and such an extraction may injure η_1.*

4. At an η_1-expansionary stage s_3, η_1 sees that

$$W_{\beta_2,s_1} \upharpoonright \gamma_{\beta_2}(p_{\eta_1})[s_1] \neq W_{\beta_2,s_3} \upharpoonright \gamma_{\beta_2}(p_{\eta_1})[s_1].$$

   Then according to the $\mathcal{I}$-strategy, η_1 extracts $\gamma_{\beta_2}(p_{\eta_1})[s_1]$ from D to recover the computation $\Phi^{B,D}_{\eta_1}(y_{1,s_1})$ to $\Phi^{B,D}_{\eta_1}(y_{1,s_1})[s_1]$ to complete cycle i_{η_1}. But, to fully recover the computation $\Phi^{B,D}_{\eta_1}(y_{1,s_1})$ to the one at stage s_1, it also needs to reenumerate the number $\gamma_{\beta_1}(p_{\eta_2})[s_0]$ into D, which is not allowed as we are constructing D as a d.c.e. set.

To get around this problem, we apply the idea given in the interactions of $\mathcal{I}$-strategies to the action done at stage s_2, to prevent the trouble described at stage s_3 from occurring. That is, when η_2 extracts $\gamma_{\beta_1}(p_{\eta_2})[s_0]$ from D, it also enumerates s_0, the stage at which cycle i_{η_2} starts an attack on β_1, into B. This enumeration undefines $\Delta^B_{\eta_1\beta_2}(z)$, which is defined at stage s_1. This enumeration will not affect η_1's purpose of completing cycle i_{η_2}, as this number is bigger than $\varphi_{\eta_2}(B, D; y_{2,s_0})[s_0]$. Note that the enumeration of s_0 into B allows us to cancel legally all other cycles, which are started after stage s_0.

With all these features considered, η acts in a more general way, by running several cycles, with a few of modifications added. Especially, $i2 - j2$ works as follows:

$i2 - j2$. (Completion on β_j)

At an η-expansionary stage $s_2 > s_1$, η sees a W_{β_j}-change below $\gamma_{\beta_j}(p_\eta)[s_1]$, then η extracts all the numbers, including $\gamma_\beta(p_\eta)[s_1]$, and all numbers being enumerated into D by strategies below outcome g_j, out of D to recover the computation $\Phi^{B,D}_\eta(y)$ to the one at stage s, where $s < s'$ is the stage at which $\Theta^B_\eta(y)[s']$ is defined and y is the witness chosen by cycle i at stage s'. At the same time, η puts s^1_{n-j} into B to undefine all $\Delta^B_{\eta'\beta'}(z)$ defined after stage s^1_{n-j} with $\eta' \subset \eta$. *Note that this enumeration into B cancels all cycles of η' started after stage s^1_{n-j}.* Cycle i passes the threshold p_η for β_j.

As pointed out above, for any z, $\Delta^B_{\eta\beta_j}(z)$ can be undefined by those η-strategies below the outcome g_j. It will not be a problem, as when $\Delta^B_{\eta\beta_j}(z)$ is defined, there are only finitely many $\mathcal{I}$-strategies below the outcome g_j having been visited, and only these strategies can undefine $\Delta^B_{\eta\beta_j}(z)$. So as discussed in the part of the interactions between $\mathcal{I}$-strategies, $\Delta^B_{\eta\beta_j}(z)$ will be eventually defined. Therefore, if η has g_j as its true outcome, then $\Delta^B_{\eta\beta_j}$ is fully defined, and computes W_{β_j} correctly and β_j is satisfied at η.

5.4 Construction

First we assign the priority of all the requirements as follows

$$\mathcal{R}_0 < \mathcal{I}_0 < \mathcal{P}_0 < \mathcal{R}_1 < \mathcal{I}_1 < \mathcal{P}_1 < \cdots < \mathcal{R}_e < \mathcal{I}_e < \mathcal{P}_e < \ldots .$$

The priority tree T is built by recursion. The top node on T is labelled $\mathcal{R}_0$. Assume that τ is a node on T. If τ is labelled $\mathcal{R}_e$, then there is one edge leaving τ labelled 1. If τ is labelled $\mathcal{P}_e$, then there are two edges leaving τ labelled $w <_L d$. It is complicated when τ is labelled $\mathcal{I}_e$. Let σ be a finite path in T starting from the top node and ending with node τ. We say that a node β on σ labelled $\mathcal{R}_j$ is *active* at τ if for every node $\eta \supset \beta$ in σ there is no edge $g_{\beta'}$ leaving η for some $\beta' \subset \beta$. A node β in σ labelled $\mathcal{R}_j$ is *satisfied* by $\eta \supset \beta$ in σ if it is active at τ and $\eta^\frown g_\beta \in \sigma$. We say $\zeta \in \sigma$ is *injured* by $\eta \supset \zeta$ in σ if there is an $\beta \subset \zeta$ in σ such that $\eta^\frown g_\beta \in \sigma$. Now if τ is labelled $\mathcal{I}_e$, then the edges leaving τ are labelled $\infty <_L f <_L d$ or additionally with g_{β_j} if there exists $\beta_j \subset \tau$ labelled $\mathcal{R}_j$, active at τ and not satisfied on τ. Their priority is ordered as follows: all g_{β_j} are to the left of ∞ and $g_{\beta_j} <_L g_{\beta_{j'}}$ if $\beta_j \subset \beta_{j'} \subseteq \tau$.

Given a path σ in T with the ending node τ, we say that a requirement R is *represented* by ζ at τ if ζ is a node in σ such that one the following conditions is true:

(a) R is $\mathcal{P}_e$. ζ is labelled $\mathcal{R}_e$ and is not injured by any $\eta \in \sigma$. Moreover, the edge leaving ζ is labelled either w or d;

(b) R is $\mathcal{I}_e$. ζ is labelled $\mathcal{I}_e$ and is not injured by any $\eta \in \sigma$. Furthermore, the edge leaving ζ is labelled either ∞, f or d.

(c) R is $\mathcal{R}_e$. $\mathcal{R}_e$ is active at τ, ζ is labelled $\mathcal{R}_e$ and is not injured by any $\eta \in \sigma$.

Continuing with the definition of T by recursion, if σ is a finite path in T which ends with a node τ then τ is labelled with the highest requirement which is not represented by any α at τ.

By a standard argument, it is easy to see that the following two propositions are true.

Proposition 5.1. *Given an infinite path P in T. For each requirement $\mathcal{R}_e$, there is strategy β in P such that β is active (i.e., not injured) in $P \upharpoonright n$ for all $n > |\beta|$.*

Proposition 5.2. *Given an infinite path P in T. For any requirement U, U is either $\mathcal{P}_e$, $\mathcal{I}_e$ or $\mathcal{R}_e$ for some e, there is a node γ on P such that for all $n > |\gamma|$, U is always represented by γ in $P \upharpoonright n$.*

Let τ be a node on T. If $\tau = \alpha$ is labelled $\mathcal{P}_e$ then it will perform a Friedberg-Muchnik's diagonalization argument. If $\tau = \beta$ is labelled $\mathcal{R}_e$ then it only defines a p.r. functional Γ_β, targeting to make $K = \Gamma_\beta^{B,D,W_e}$. If $\tau = \eta$ is labelled $\mathcal{I}_e$, then

- η will have a threshold p_η. In the construction, p_η can be cancelled only when η is initialized. So, if η is on the true path, then η can be initialized only finitely often, which ensures that p_η is finite.
- η will run perhaps infinitely many cycles, each of which will build a p.r. functional Θ_η^B at η-expansionary stages, targeting to make $W_\eta = \Theta_\eta^B$.
- If at cycle i, η sees $\Theta_\eta^B(y) \neq W_\eta(y)$ for some (least) y with $\Theta_\eta^B(y)$ defined, then cycle i will start an attack via this y, trying to make the associated computation clear of those $\mathcal{R}$-strategies active at η.

At a stage s, define the length of agreement function $l(\eta, s)$ as

$$l(\eta, s) = \max\{x \mid \forall y < x(W_\eta \upharpoonright y[s] = \Phi_\eta^{B,D} \upharpoonright y[s])\},$$

where $\Phi_\eta^{B,D} \upharpoonright y[s]$ are η-*believable computations* at stage s, and the maximal length function $m(\eta, s)$ as

$$m(\eta, s) = \max\{l(\eta, t), 0 \mid t < s \text{ and } t \text{ is an } \eta\text{-stage}\}.$$

Say that s is an η-*expansionary stage* if s is an η-stage and $l(\eta, s) > m(\eta, s)$.

Also we assume that at each stage $s > 0$, exactly one number, k_s, is enumerated into K.

We now describe the full construction. At stage s, we first define the current true path σ_s with $|\sigma_s| \leq s$, and we call the nodes on σ_s *accessible* at stage s, and then take actions on the accessible nodes.

Stage 0**:** Initialize all the nodes on T and let $A = B = D = \emptyset$ and all constructed functionals be $\emptyset$. Let $\sigma_0 = \lambda$, the root of T. Go to stage 1.

Stage $s > 0$**:** Stage s consists of two phases:

Phase 1. Let k_s be the element enumerated into K at stage s and reset all $\mathcal{I}$-strategies η with the threshold $p_\eta \geq k_s$. If η is reset and τ is a strategy with lower priority, then τ is initialized automatically.

Phase 2. Define σ_s inductively starting from λ with $|\sigma_s| \leq s$. At the end of stage s, initialize all the strategies with priority lower than σ_s.

Substage 0**:** Let $\sigma_s(0) = \lambda$. Go to substage 1.

Substage $t > 0$**:** Having $\sigma_s(t)$. If $|\sigma_s(t)| = s$ then take $\sigma_s = \sigma_s(t)$ and go to the next stage. If $|\sigma_s(t)| < s$, then find the outcome of $\sigma_s(t)$ as follows:

Case 1: $\sigma_s(t) = \alpha$ is a $\mathcal{P}_e$-strategy. There are three subcases.

$\alpha 1$. x_α is not defined. Then α defines it as a fresh number and let $\sigma_s = \sigma_s(t)$ and go to the next stage.

$\alpha 2$. x_α is defined and α is not satisfied yet. Check whether

$$\Phi_\alpha^B(x_\alpha)[s] \downarrow = 0.$$

If yes, then put x_α into A, declare that α is satisfied at stage s and let $\sigma_s = \sigma_s(t)$. Otherwise, let $\sigma_s(t+1) = \alpha^\frown w$ and go to the next substage.

$\alpha 3$. If α has been satisfied before and from then on, it has not been initialized, then let $\sigma_s(t+1) = \alpha^\frown d$ and go to the next substage.

Case 2: $\zeta = \beta$ is an $\mathcal{R}_e$-strategy.

Check whether there is $k \in K_s$ but $\Gamma_\beta^{B,D,W_e}(k)[s] = 0$. If so, take the least such k and then enumerate $\gamma_\beta(k)[s]$ into D to undefine $\Gamma_\beta^{B,D,W_e}(k')[s]$ for $k' \geq k$. Otherwise, define $\Gamma_\beta^{B,D,W_e}(k)[s] = K_s(k)$ for all $k \leq s$ and $\Gamma_\beta^{B,D,W_e}(k) \uparrow$, with use $\gamma(k)[s]$ as a fresh number, if $\gamma(k)$ is requested to be big, or equal to the previous use if no such a request exists. Let $\sigma_s(t+1) = \sigma_s(t)^\frown 1$ and go to the next substage.

Case 3: $\sigma_s(t) = \eta$ is an $\mathcal{I}_e$-strategy. If p_η is not defined, then define it as a fresh number and let η start cycle 0. Let $\sigma_s = \sigma_s(t)^\frown f$ and go to the next stage. Otherwise, suppose that cycle i is the greatest cycle active at stage s. There are three cases.

η1. If η is satisfied then let $\sigma_s(t+1) = \sigma_s(t)^\frown d$, and go to the next substage.

η2. If η1 fails and s is not an η-expansionary stage, then let $\sigma_s(t+1) = \sigma_s(t)^\frown f$, and go to the next substage.

η3. If η1 fails and s is an η-expansionary stage, then see whether some *active cycle* $i' < i$ can complete its current attack on some β_j, with $0 \le j \le n$. Here $\beta_0 \subset \beta_1 \subset \cdots \subset \beta_n \subset \eta$ is a list of $\mathcal{R}$-strategies active at η.

η3.1. If *yes*, then let the least such active cycle i' to do so. According to the value of j, the step at which cycle i' is at at the moment, there are two cases.

η3.1.1. $j > 0$.

Then (1) remove from D all numbers being enumerated into D by those strategies below outcome g_j after stage s^1_{n-j}, the stage at which the j-th attack of cycle i' is started (we are completing the attack on β_j), and request that for any $k \ge p_\eta$, $\gamma_{\beta_j}(k)$ be defined as big; (2) enumerate $\gamma_{\beta_{j-1}}(p_\eta)[s]$ into D to start the attack for β_{j-1}, and request that for any $k \ge p_\eta$, $\gamma_{\beta_{j-1}}(k)$ be defined as big; also enumerate s^1_{n-j} into B to undefine all $\Delta_{\eta'\beta'}(z)$ and $\Theta_{\eta'}(z)$ defined after stage s^1_{n-j}; (3) extend the definition of $\Delta^B_{\eta\beta_{j-1}}$ up to $\gamma_{\beta_{j-1}}(p_\eta)[s]$: for $z \le \gamma_{\beta_{j-1}}(p_\eta)[s]$ such that $\Delta^B_{\eta\beta_{j-1}}(z)$ is currently not defined, define it as $W_{\beta_{j-1},s}(z)$ with use big; (4) let $\sigma_s(t+1) = \sigma_s(t)^\frown g_{j-1}$, and go to the next substage.

η3.1.2. $j = 0$.

Then (1) remove from D all numbers being enumerated into D by those strategies below outcome g_j after stage s^1_n, the stage at which the 0-th attack of cycle i' is started (we are completing the attack on β_0) and request that for any $k \geq p_\eta$, $\gamma_{\beta_0}(k)$ be defined as big; (2) remove from D all numbers being enumerated into D by those strategies below outcome ∞ between the stage s_0 at which $\Theta^B_\eta(y)$ is defined and the stage s^1_0, the stage at which the n-th attack of cycle i' is started; (3) enumerate s_0 into B to undefine those $\Delta_{\eta'\beta'}(z)$ defined after stage s_0. Declare that η is satisfied at stage s, and define $\sigma_s = \eta$. Go to the next stage.

η3.2. If *no*, then see whether $\Theta^B_\eta[s]$ and $W_{\eta,s}$ agree on all the numbers in Θ^B_η's domain.

η3.2.1. If no, then cycle i starts its attack for β_n, by enumerating $\gamma_{\beta_n}(p_\eta)[s]$ into D and extending the definition of $\Delta^B_{\eta\beta_n}$ to numbers up to $\gamma_{\beta_n}(p_\eta)[s]$: for $z \leq \gamma_{\beta_n}(p_\eta)[s]$ such that $\Delta^B_{\eta\beta_n}(z)$ is currently not defined, define it as $W_{\beta_n,s}(z)$ with use big. Request that for any $k \geq p_\eta$, $\gamma_{\beta_n}(k)$ be defined as big. Let $\sigma_s(t+1) = \sigma_s(t)^\frown g_n$ and go to the next substage.

η3.2.2. If yes, then extend the definition of Θ^B_η up to $l(\eta, s)$: for $z < l(\eta, s)$ such that $\Theta^B_\eta(z)$ is currently not defined, define is as $W_{\eta,s}(z)$, with use big if $\Theta^B_\eta(z)$ has not been defined before, or $\Phi^{B,D}_\eta(z)$ has a new computation, or with use the same as before it $\Theta^B_\eta(z)$ has definition before, and the computation $\Phi^{B,D}_\eta(z)$ is the same as before. Let $\sigma_s(t+1) = \sigma_s(t)^\frown\infty$ and go to the next substage.

This completes the full construction.

5.5 Verification

Let $TP = \liminf_s \sigma_s$ be the true path of the construction. We first prove that TP is infinite and then show that all the requirements are satisfied along the true path.

Lemma 5.3. *For each $\zeta \in TP$,*

(i) ζ can be reset or initialized at most finitely often,

(ii) there is an outcome O of ζ such that $\zeta^\frown O \in TP$, and

(iii) ζ can initialize $\zeta^\frown O$ at most finitely often.

Therefore, TP is infinite.

Proof. We prove this lemma by induction on the length of strategies on the true path. Let ζ^- be the immediate predecessor of ζ. By the assumption that the (i)-(iii) are all true for ζ^-, there is a least stage s_{ζ^-} after which (i) ζ^- cannot be reset or initialized, (ii) no nodes on the left of ζ can be visited (again), (iii) ζ can be initialized by ζ^- at most finitely many times.

We now prove that the lemma is true for ζ. From the assumption on ζ^- and the choice of s_0, after stage s_0, ζ can never be initialized by strategies with higher priority. Let $s_1 \geq s_0$ be the stage at which p_ζ is selected (if ζ is an $\mathcal{I}$-strategy). Then ζ can be reset only when some number $k \leq p_\zeta$ enters K, which can happen at most $p_\zeta + 1$ many times. This shows that (i) is true for ζ.

We now show that (ii) is true for ζ. There are three cases.

Case 1. *$\zeta = \alpha$ is a $\mathcal{P}$-strategy.*

By the choice of s_0, when α is first visited after s_0, then ζ defines x_α. This x_α will not be cancelled in the remainder of the construction. If $\Phi_\alpha^B(x_\alpha)$ never converges to 0, then any further α-stage will be an $\alpha^\frown w$-stage, and hence according to the definition of TP, $\alpha^\frown w$ is on TP. If there is a stage $s_2 > s_0$ at which $\Phi_\alpha^B(x_\alpha)$ converges to 0, then at this stage, x_α is enumerated into A, and also all strategies with lower priority are initialized at the same time. Note that if later, a strategy with higher priority enumerates a number into B, then this strategy will have a new outcome and a strategy on the left of ζ^- would be visited at this stage, which contradicts the assumption on s_0. This means that the computation $\Phi_\alpha^B(x_\alpha)$ is protected from all the enumerations of other strategies. Therefore, after this stage, any further α-stage will be an $\alpha^\frown d$-stage, and $\alpha^\frown d$ is on TP. (ii) is true for α. This also shows that (iii) is true for α as α initializes the other strategies by at most once after stage s_0.

Case 2. *$\zeta = \beta$ is an $\mathcal{R}$-strategy.*

As β has only one outcome, and any β-stage is also a $\beta^\frown 1$-stage, $\beta^\frown 1$ is on TP. (iii) is also true for β as it does no initialization at all in the construction.

Case 3. *$\zeta = \eta$ is an $\mathcal{I}$-strategy.*

In the construction, η can run perhaps infinitely many cycles. One case is that η only runs finitely many cycles. In this case, η will not have the Σ_3 outcome g_j as its true outcome, which needs η to run infinitely many cycles. Let i be the greatest cycle active in the remainder of the construction. Then either after starting this cycle, at all η-expansionary stages, Θ_η^B always agrees with W_η on all arguments, or η sees a disagreement between Θ_η^B and W_η at some y and the associated computation $\Phi_\eta^{B,D}(y)$ eventually passes the threshold p_η for all $\mathcal{R}$-strategies active at η, and eventually a disagreement between Θ_η^B and W_η is created and preserved. In the latter case, any further η-stage is also an $\eta^\frown d$-stage, while in the first case, if there are only finitely many η-expansionary stages, then after a stage large enough, any further η-stage is also an $\eta^\frown f$-stage, and if there are infinitely many η-expansionary stages, then any such a stage is also an $\eta^\frown\infty$-stage. In all cases, η has an outcome O with $\eta^\frown O$ on the true path.

We now consider the other case: η runs infinitely many cycles in the whole construction. Let

$$\beta_0 \subset \beta_1 \subset \cdots \subset \beta_n \subset \eta$$

be a list of $\mathcal{R}$-strategies, then infinitely many times, η has outcomes among $g_j, 0 \leq j \leq n$, and hence, one of them (the least one), g_j say, will be visited infinitely often. According to the definition of TP, $\eta^\frown g_j$ is on TP. Thus, we have shown that (ii) is true for η. (iii) follows from (ii) easily as $\eta^\frown O$ is initialized only when some outcome on the left of O turns out to be true, which can happen only finitely many times, by (ii).

This shows that (i)-(iii) is true for every node on the true path. Therefore, TP is infinite. This completes the proof of Lemma 5.3. □

Note that the proof of Lemma 5.3 also shows that all the $\mathcal{P}$-strategies are satisfied along the true path.

Lemma 5.4. *Given a requirement $\mathcal{P}_e$, let α be the last $\mathcal{P}_e$-strategy on TP. Then the requirement $\mathcal{P}_e$ is satisfied by α.*

Now we prove that all the $\mathcal{I}$-strategies are satisfied along the true path.

Lemma 5.5. *Given a requirement $\mathcal{I}_e$, let η be the last $\mathcal{I}_e$-strategy on TP. Then the requirement $\mathcal{I}_e$ is satisfied by η.*

Proof. By the choice of η, it must be the situation that either $\eta^\frown\infty \in TP$, $\eta^\frown f \in TP$ or $\eta^\frown d \in TP$, since otherwise there is a strategy $\eta' \supset \eta$ in TP such that $\mathcal{I}_e$ is represented by η' again, which contradicts the choice of η. Let s be the least stage after which η can neither be initialized nor be reset. Let i be the greatest cycle run by η that is active in the remainder of the construction. Then as classified in the proof of Lemma 5.3, this cycle can have one of f, ∞ and d as its true outcome, O say, such that $\eta^\frown O$ is on TP.

Case 1. $\eta^\frown f \in TP$.

In this case, there are only finitely many η-expansionary stages. Let $s' > s$ be the largest η-expansionary stage. Then it is easy to see that there is a $y \leq l(\eta, s')$ such that $W_\eta(y) \neq \Phi_\eta^{B,D}(y)$, and $\mathcal{I}_e$ is satisfied vacuously.

Case 2. $\eta^\frown\infty \in TP$.

In this case, there are infinitely many η-expansionary stages and cycle i sees no disagreement between Θ_η^B and W_η, which means that cycle i never starts to attack during the construction. Let s' be the stage at which cycle i was started for the last time. Suppose that $\Phi_\eta^{B,D} = W_\eta$, then $\Phi_\eta^{B,D}$ is total. We need to show that Θ_η^B is well-defined and computes W_η correctly. The latter is obviously true by the assumption on i. We only need to show that Θ_η^B is defined as total.

Fix z. Let $s'' \geq s'$ be the least stage such that for all stages $t \geq s''$, the computation $\Phi_\eta^{B,D}(z)[t]$ is identical to the computation $\Phi_\eta^{B,D}(z)[s'']$. Let s''' be the first η-expansionary stage after stage s''. W.l.o.g., suppose that η defines $\Theta_\eta^B(z)$ at this stage, at which only finitely many strategies below the outcome ∞ are visited at this stage. Then $\Theta_\eta^B(z)$ can only be undefined by these strategies, when they moved numbers from D, and these are enumerated into D before stage s'', and whenever

$\Theta^B_\eta(z)$ is redefined again, the use $\theta_\eta(z)$ is again defined as s'' (no increment as the computation $\Phi^{B,D}_\eta(z)$ remains the same). Thus, after a stage such that no numbers being enumerated into D before stage s'' are extracted from D, $\Theta^B_\eta(z)$ cannot be undefined anymore, which means that $\Theta^B_\eta(z)$ is defined.

Therefore, Θ^B_η is total, and η is satisfied.

Case 3. $\eta^\frown d \in TP$.

Let y be the least disagreement between Θ^B_η and W_η that cycle i sees. Then cycle i starts attacks to ensure that the associated computation $\Phi^{B,D}_\eta(y)$ is clear of the γ_{β_j}-uses for $0 \leq j \leq n$. As $\eta^\frown d$ is on the true path, we know that eventually, cycle i completes each attack by having a corresponding W's change, and at stage s' say, when cycle i completes its attack for β_0, it recovers the computation $\Phi^{B,D}_\eta(y)$ to the one from a previous stage to create a disagreement between $\Phi^{B,D}_\eta(y)$ (which is equal to 0) and $W_\eta(y)$ (which is equal to 1 now). As W_η is c.e., $\Phi^{B,D}_\eta(y)$ is now preserved from lower priority strategies (they are initialized at stage s') and also higher priority strategies (all the $\mathcal{R}$-strategies active at η now have their γ-uses bigger than the use $\varphi_\eta(B, D; y)$). Thus the requirement $\mathcal{I}_e$ is satisfied by η through a successful isolation strategy. □

Now we show that every $\mathcal{R}$-strategy is satisfied along the true path.

Lemma 5.6. *Suppose that $\eta \in TP$ is active on all $TP \upharpoonright n$ for $n > |\eta|$ and $\eta^\frown g_\beta \in TP$. Then $W_\beta = \Delta^B_{\eta\beta}$.*

Proof. As η has g_β as its true outcome, we know that η runs infinitely many cycles in the construction, and infinitely many of them start attacks on β but cannot complete these attacks. This means that $\Delta^B_{\eta\beta}$ is defined infinitely often, and W_β always agree with $\Delta^B_{\eta\beta}$ on all arguments z, if $\Delta^B_{\eta\beta}(z)$ is defined. Here we have a stage large enough such that no outcome of η on the left of g_β can be true afterwards.

We claim that $\Delta^B_{\eta\beta}$ is defined on each argument z. That is, $\Delta^B_{\eta\beta}(z)$ can be undefined, and hence redefined, at most finitely often for z. Suppose that cycle i defines $\Delta^B_{\eta\beta}(z)$ at stage s. Note that at this stage, only finitely many strategies below the outcome g_β are visited and $\Delta^B_{\eta\beta}(z)$ can only be undefined by these strategies, when they moved numbers from D, and these numbers are enumerated into D before stage s. Thus, after a finitely many times, after no numbers being enumerated into D before stage s are extracted from D, and hence $\Delta^B_{\eta\beta}(z)$ cannot be undefined anymore. This ensures that $\Delta^B_{\eta\beta}(z)$ is defined.

Therefore, $\Delta^B_{\eta\beta}$ is totally defined. Obviously, $\Delta^B_{\eta\beta}$ computes W_β correctly, as otherwise, some node on the left of $\eta^\frown g_\beta$ will be visited, contradicting our assumption that no outcome of η on the left of g_β can be true. □

Lemma 5.7. *Given a requirement $\mathcal{R}_e$ for $e \in \omega$, let β be the last $\mathcal{R}_e$-strategy on TP.*

(i) If $\Gamma_\beta^{B,D,W_\beta}$ is defined totally, then $K = \Gamma_\beta^{B,D,W_\beta}$. So the requirement $\mathcal{R}_e$ is satisfied by β.

(ii) If $\Gamma_\beta^{B,D,W_\beta}$ is not total, then there is an $\mathcal{I}$-strategy $\eta \supset \beta$ such that $\eta^\frown g_\beta \in TP$. Thus, the requirement $\mathcal{R}_e$ is satisfied by η via $W_\beta = \Delta^B_{\eta\beta}$.

Proof. (i) Suppose that $\Gamma_\beta^{B,D,W_\beta}$ is defined totally, then by our construction, for any $z \in \omega$, $K(z) = \Gamma_\beta^{B,D,W_\beta}(z)$, and therefore, $\mathcal{R}_e$ is satisfied by β via $K = \Gamma_\beta^{B,D,W_\beta}$.

(ii) If $\Gamma_\beta^{B,D,W_\beta}$ is not total, i.e., there is a least $z \in \omega$ such that $\Delta^B_{\eta\beta}(z)$, for some z, is undefined infinitely often in the construction. Then this z is a threshold of some η-strategy below β, and this η runs infinitely many cycles, with infinitely many of them starting attacks on β but none of them can complete these attacks. That is, as shown in Lemma 5.6, $\Delta^B_{\eta\beta}$ is totally defined, and computes W_β correctly. Therefore, the requirement $\mathcal{R}_e$ is satisfied by η via $W_\beta = \Delta^B_{\eta\beta}$. □

This completes the proof of Theorem 5. ◇◇

Chapter 6

DIAMOND IN THE C.E. TRUTH-TABLE DEGREES

6.1 Introduction

In the previous chapters, we investigate the c.e. sets under the Turing reduction and consider their variants in Ershov hierarchy. Now, we study the c.e. sets from another aspect by considering their behaviors under a stronger reducibility — the truth-table reduction. Similar to the investigation on the interactions between $\mathbf{R}$ and $\mathbf{D}_2$, it is natural to compare the structural differences between $\mathbf{R}$ and $\mathbf{R}_{tt}$.

In 1966, Lachlan proved that no diamond preserving both 0 and 1 can be embedded into the c.e. Turing degrees [46]. However, Cooper [13] showed that such a diamond can be embedded into the Δ_2^0 degrees if we do not require the atoms to be c.e. degrees. Later, Epstein [26] showed that those two atoms can be made to be both low or both high, and Downey proved in [25] that both atoms can even be d.c.e. degrees, giving an extremely sharp result in terms of the Ershov hierarchy.

Since the proof of Lachlan's Non-Diamond Theorem is valid in the setting of the c.e. *wtt*-degrees, no such diamond exists in the c.e. *wtt*-degrees. However, Jockusch and Mohrherr showed in [43] that the diamond lattice can be embedded into the c.e. *tt*-degrees preserving 0 and 1 and, furthermore, they pointed out that the two atoms can be low. In this chapter, we show that such a diamond can be embedded into the c.e. tt-degrees in a way such that both atoms are superhigh.

Theorem 6 (Cenzer, Franklin, Liu and Wu [10]) *There are superhigh computably enumerable sets A and B such that $\mathbf{0}_{tt}$, $deg_{tt}(A)$, $deg_{tt}(B)$, and $\mathbf{0}'_{tt}$ form a diamond in the computably enumerable tt-degrees.*

Our construction differs from Jockusch and Mohrherr's in several important ways. Jockusch and Mohrherr's construction involves only a finite injury argument, while ours involves an infinite injury argument. This is necessary to make A and B superhigh. Due to this, our sets A and B will not have some of the nice properties that Jockusch and Mohrherr's do. For instance, they were able to build their atoms A and B with $A \cup B = K$, guaranteeing that $K \equiv_{tt} A \cup B$ in a very obvious way. In our construction, the superhighness strategies will force us to enumerate elements into A and B from time to time to maintain our computations that witness $A' \geq_{tt} TOT$ and $B' \geq_{tt} TOT$. To ensure that $K \leq_{tt} A \oplus B$, we dedicate the numbers of the form $\langle x, 0\rangle$ to meeting this requirement. This allows us to replace Jockusch and Mohrherr's conclusion that $x \in K$ if and only if $x \in A \cup B$ by the slightly more complicated conclusion that $x \in K$ if and only if $\langle x, 0\rangle \in A \cup B$. Again, for the consistency between the superhighness strategies and the minimal pair strategies, we need to be extremely careful when we switch from one outcome to another one. When we choose a *fresh* number as a γ-use or a δ-use at stage s, this number is the least number bigger than the corresponding restraint that is not of the form $\langle x, 0\rangle$.

6.2 Requirements and basic strategies

To prove Theorem 6, we will construct two c.e. sets A and B such that both of them are superhigh, K is truth-table reducible to $A \oplus B$, and the tt-degrees of A and B form a minimal pair in the tt-degrees. A and B will satisfy the following requirements:

$\mathcal{P}$: $K \leq_{tt} A \oplus B$;

$\mathcal{S}^A$: $TOT \leq_{tt} A'$;

$\mathcal{S}^B$: $TOT \leq_{tt} B'$;

$\mathcal{N}_{i,j}$: $[i]^A = [j]^B = g$ total $\Rightarrow g$ is computable;

Recall that $TOT = \{e : \varphi_e \text{ is total}\}$ is a Π^0_2-complete set. Therefore, if $\mathcal{S}^A$ and $\mathcal{S}^B$ are satisfied, then A and B will both be superhigh.

6.2.1 The $\mathcal{P}$-Strategy

To satisfy the requirement $\mathcal{P}$, we simply code K into $A \oplus B$. We will use a computable enumeration of K such that at each odd stage s, exactly one number, k_s, enters K. At each odd stage s, we will enumerate $\langle k_s, 0\rangle$ into A, B, or both. We will decide which of these sets to enumerate $\langle k_s, 0\rangle$ into based on the actions of the minimal pair strategies $\mathcal{N}_{i,j}$. If $k \notin K$, then $\langle k, 0\rangle$ will never be enumerated into A or B. It is obvious that we will have the equality $K = \{k : \langle k, 0\rangle \in A \cup B\}$, and hence $K \leq_{tt} A \oplus B$.

The $\mathcal{P}$-requirement is global, so we do not put its outcome on the construction tree.

6.2.2 An $\mathcal{N}_{i,j}$-Strategy

Recall that if $[i]$ is a tt-reduction, then for any oracle $X \subseteq \omega$ and any input x, $[i]^X(x)$ converges. The computation $[i]^X(x)$ can be injured at most finitely many times due to the enumeration of numbers less than or equal to $|\tau_{\varphi_i(x)}|$ into X in our construction.

For the requirement $\mathcal{N}_{i,j}$, we apply the diagonalization argument introduced by Jockusch and Mohrherr in [43]. That is, once we see a disagreement between $[i]^A$ and $[j]^B$, we will preserve it forever to make $[i]^A \neq [j]^B$. On the other hand, if $[i]^A$ and $[j]^B$ are equal and total, then we will ensure that they are computable.

Given values for A_s and B_s at stage s, we will define A_{s+1} and B_{s+1} at stage $s+1$ by enumerating more elements into them. Furthermore, if we know that $[i]^A$ and $[j]^B$ differ at k at stage s, we will describe a way of preserving this disagreement at stage $s+1$ even though the enumeration of numbers into A or B (or both) might change the computations involved. Let n be a number we want to put into $A_{s+1} \cup B_{s+1}$. There are two cases.

(1) Our number n is of the form $\langle x, 0\rangle$ for some x. Then n is enumerated into A, B, or both for the sake of the requirement $\mathcal{P}$. There are three subcases.

Subcase 1: If $[i]^{A_s}(k) = [i]^{A_s \cup \{n\}}(k)$, then n is enumerated into A but not into B. Both values are preserved, and the disagreement is preserved as well.

Subcase 2: If Subcase 1 does not apply but $[j]^{B_s}(k) = [j]^{B_s \cup \{n\}}(k)$, then n is enumerated into B but not into A. As in Case 1, the disagreement is preserved.

Subcase 3: If $[i]^{A_s}(k) \neq [i]^{A_s \cup \{n\}}(k)$ and $[j]^{B_s}(k) \neq [j]^{B_s \cup \{n\}}(k)$, then n is enumerated into both A and B. In this case, the disagreement is again preserved, as both values are changed.

Note that once one subcase above applies, then we initialize all the strategies with lower priority to avoid the conflict among the $\mathcal{N}$-strategies — obviously, such initializations can happen at most finitely often. We need to be careful here when more $\mathcal{N}$-strategies are considered. It can happen that if we decide to enumerate into A, B, or both, we also need to consider those $\mathcal{N}$-strategies with higher priority, say $\mathcal{N}_{i',j'}$, as we need to avoid the following situation: according to the $\mathcal{N}_{i,j}$-strategy, at stage s_1, a number n_1 is enumerated into A, and at stage s_2, a number n_2 is enumerated into B (corresponding to Subcases 1 and 2, respectively), and such enumerations change $[i']^A(m)$ and $[j']^B(m)$, though separately, and at the next $\mathcal{N}_{i',j'}$-expansionary stage, we may have $[i']^A(m) = [j']^B(m)$, which is different from its original value — $\mathcal{N}_{i',j'}$ is injured.

With this in mind, when we see that the $\mathcal{P}$-strategy wants to enumerate a number into A, B, or both and that an $\mathcal{N}_{i,j}$-strategy is attempting to preserve a disagreement that already exists, instead of automatically implementing the enumeration, we first check whether such an enumeration into A can lead to a disagreement between $[i']^A$ and $[j']^B$. If not, then we just work as described above (in Subcase 3, we now enumerate n into B and check whether this enumeration into B can lead to a disagreement for $\mathcal{N}_{i',j'}$ — here n is enumerated into A and B separately). Otherwise, we start to preserve this disagreement to satisfy $\mathcal{N}_{i',j'}$ — the $\mathcal{N}_{i,j}$ considered above is initialized, and again, even if Subcase 3 applies, we do not enumerate n into B.

(2) Our number n is a number chosen by an $\mathcal{S}_e^A$-strategy or an $\mathcal{S}_e^B$-strategy. Without loss of generality, suppose that n is selected by an $\mathcal{S}_e^A$-strategy and we want to put it into A. As in the standard construction of high sets, we will consider "believable" computations to allow us to handle the potential infinitary injury to the negative strategies caused by the higher priority highness strategies; for instance, $[i]^A(m)$. In this way, when we see $[i]^A$ and $[j]^B$, if this $\mathcal{S}_e^A$-strategy has higher priority than $\mathcal{N}_{i,j}$, then the enumeration of n into A does not affect the computation $[i]^A(m)$. This will be described further in the discussion of the $\mathcal{S}_e^A$-strategies below. The notion of a believable computation will be defined formally in Definition 6.1.

An $\mathcal{N}_{i,j}$-strategy has three outcomes: ∞, f and d, where ∞ denotes that there are infinitely many expansionary stages, f denotes that there are only finitely many expansionary stages, but no disagreement is produced, and d denotes that a disagreement between $[i]^A$ and $[j]^B$ is produced and preserved successfully.

6.2.3 An $\mathcal{S}_e^A$-Strategy

To make A superhigh, instead of giving a truth-table reduction from TOT to A' explicitly, we will construct a binary functional $\Gamma^A(e,x)$ such that for all $e \in \omega$,

$$\mathrm{TOT}(e) = \lim_{x\to\infty} \Gamma^A(e,x)$$

with $|\{x : \Gamma^A(e,x) \neq \Gamma^A(e,x+1)\}|$ bounded by a computable function h, which will ensure that $\mathrm{TOT} \leq_{tt} A'$. The relativized Limit Lemma will guarantee that A will be superhigh. (In the case of B, we will construct a binary functional $\Delta^B(e,y)$ satisfying a similar requirement.) The crucial point is to find this computable bounding function h.

As usual, $\mathcal{S}^A$ is divided into infinitely many substrategies $\mathcal{S}_e^A$, $e \in \omega$, each of which has two outcomes, ∞ (a Π_2^0-outcome) and f (a Σ_2^0-outcome), where ∞ denotes the guess that φ_e is total and f denotes the guess that φ_e is not total.

Let β be an $\mathcal{S}_e^A$-strategy on the priority tree. As usual, we have the following standard definition of length agreement function:

$$l(\beta, s) = \max\{\ x < s : s \text{ is a } \beta\text{-stage and } \varphi_e(y)[s]\downarrow \quad \text{for all} \quad y < x\};$$

$$m(\beta, s) = \max\{\ l(\beta, t) : t < s \text{ is an } \beta\text{-stage}\};$$

where t is a β-stage if β is visited at stage t. Say that s is a β-expansionary stage if $s = 0$ or $l(\beta, s) > m(\beta, s)$.

Let s be a β-stage. If s is a β-expansionary stage, then we believe that φ_e is total, and we undefine every $\Gamma^A(e, x)$ defined by lower priority strategies by enumerating the corresponding $\gamma(e, x)$ into A and then define $\Gamma^A(e, y)$ to be 1 for the least y such that $\Gamma^A(e, y)$ is undefined. If s is not a β-expansionary stage, then we believe that φ_e is not total, and again, we undefine those $\Gamma^A(e, x)$ defined by lower priority strategies by enumerating the corresponding $\gamma(e, x)$ into A and then define $\Gamma^A(e, y)$ to be 0 for the least y with $\Gamma^A(e, y)$ not defined. Thus, if there are infinitely many β-expansionary stages (so φ_e is total, $e \in$ TOT, and ∞ is the true outcome of β), then $\Gamma^A(e, x)$ is defined as 1 for almost all $x \in \omega$. On the other hand, if there are only finitely many β-expansionary stages (so φ_e is not total, $e \notin$ TOT, and f is the true outcome of β), then $\Gamma^A(e, x)$ is defined as 0 for almost all $x \in \omega$.

Thus, for a fixed $\mathcal{S}_e^A$-strategy β on the construction tree, β will attempt to redefine $\Gamma^A(e, x)$ for almost all x, with the exception that (a) some γ-uses are prevented from being enumerated into A by higher priority strategies (when a disagreement is produced), or (b) $\Gamma^A(e, x)$ is defined by another $\mathcal{S}_e^A$-strategy with higher priority. In particular, if β is the $\mathcal{S}_e^A$-strategy on the true path, then there are only finitely many strategies with higher priority that can be visited during the whole construction, and hence β will succeed in defining $\Gamma^A(e, x)$ for almost all x.

Suppose that β is on the true path and n is the length of β. We will see that $|\{x : \Gamma^A(e, x) \neq \Gamma^A(e, x+1)\}| \leq 2^{3^n+1}$. To see this, note that (a) above can happen at most 2^{3^n} times, as there are at most 3^n many strategies with length less than n, and each time when one of them produces (not preserves) a disagreement, a restraint is set, preventing α from rectifying $\Gamma^A(e, x)$ for some x. Note that after an $\mathcal{N}$-strategy

α produces a disagreement, say at stage s, whenever α requires us to preserve this disagreement, all the strategies with lower priority will be initialized, and at the same time, all of the γ-uses and δ-uses defined after stage s will be enumerated into A and B respectively (one by one, as pointed out above, for the sake of the $\mathcal{N}$-strategies with priority higher than α). It is crucial for us to ensure that TOT is truth-table reducible to A' and B', as we will discuss below. Here, when β is initialized by a strategy with higher priority with length $\geq n$, an $\mathcal{S}_e^A$-strategy β' on the left of β is visited, and β' takes the responsibility of rectifying $\Gamma^A(e,x)$ for some x, which can lead to an inequality between $\Gamma^A(e,x)$ and $\Gamma^A(e,x+1)$. Thus, (b) can happen at most 3^n many times. In total, the number of those x such that β cannot rectify $\Gamma^A(e,x)$ is at most 2^{3^n+1}, which ensures that $Tot \leq_{tt} A'$, where the corresponding bounding function h is given by $h(e) = 2^{3^e+1}$.

We have seen some interactions between the $\mathcal{P}$-strategy and the $\mathcal{N}$-strategies. Now we describe the interactions between the $\mathcal{N}$-strategies, the $\mathcal{S}$-strategies, and the $\mathcal{P}$-strategy.

Assume that α is an $\mathcal{N}_{i,j}$-strategy, β is an $\mathcal{S}_e^A$-strategy, and ζ is an $\mathcal{S}_{e'}^B$-strategy with $\beta^\frown\infty \subseteq \zeta^\frown\infty \subseteq \alpha$. The following may happen: at a stage s, a disagreement between $[i]^A$ and $[j]^B$ appears at α, so α wants to preserve this disagreement by initializing all of strategies with lower priority. However, this disagreement can be destroyed by β and ζ, as they may enumerate small γ-uses and δ-uses into A and B separately. To avoid this, we require only that α recognizes α-believable computations, defined formally below.

Definition 6.1. *Let α be an $\mathcal{N}_{i,j}$-strategy, and β be an $\mathcal{S}_e^A$-strategy with $\beta^\frown\infty \subseteq \alpha$.*

(1) A computation $[i]^{A_s}(m)$ is α-believable at β at stage s if for each x with $\gamma(e,x)[s]$ defined by β and less than the length of the truth-table of $[i](m)$, $\Gamma^{A_s}(e,x)[s]$ is equal to 1.

(2) A computation $[i]^{A_s}(m)$ is α-believable at stage s if it is α-believable at β at stage s for any $\mathcal{S}_e^A$-strategy β, $e \in \omega$, with $\beta^\frown\infty \subseteq \alpha$.

We can define an α-believable computation $[j]^{B_s}(m)$ similarly.

We are ready to define an α-expansionary stage for an $\mathcal{N}_{i,j}$-strategy α.

Definition 6.2. *Let α be an $\mathcal{N}_{i,j}$-strategy. The length of agreement between $[i]^A$ and $[j]^B$ is defined as follows:*

$$l(\alpha, s) = \max\{x < s : \textit{for all } y < x,\ [i]^A(y)[s] = [j]^B(y)[s]$$

$$\textit{via } \alpha\textit{-believable computations}\}.$$

$$m(\alpha, s) = \max\{l(\alpha, t) : t < s \textit{ is an } \alpha\textit{-stage}\}.$$

Say that a stage s is α-expansionary if $s = 0$ or $l(\alpha, s) > m(\alpha, s)$.

Now we consider the situation when β, an $\mathcal{S}_e^A$-strategy, changes its outcome from f to ∞ at a β-expansionary stage. Let s' be the last β-expansionary stage. Unlike the construction of high degrees, to make A and B superhigh, we need to enumerate all the γ-uses and δ-uses defined by strategies below outcome f between stages s' and s into A and B. This enumeration also takes place when the outcome of an $N_{i,j}$ node moves from f to ∞. Again, these numbers cannot be enumerated into A and B simultaneously, as discussed above in the section on $\mathcal{N}$-strategies, for the sake of $\mathcal{N}$-strategies with priority higher than β. Let F^A_β and F^B_β be the collections of these γ-uses and δ-uses respectively. We put the numbers in $F^A_\beta \cup F^B_\beta$ into A or B correspondingly, one by one, from the least to the greatest, and whenever one number is enumerated, we reconsider the $\mathcal{N}$-strategies with higher priority to see whether a disagreement appears. Once such a disagreement appears at an $\mathcal{N}$-strategy, say α, we stop the enumeration as we need to satisfy α via this disagreement. In this case, β is injured. Note that β can be injured in this way only by those $\mathcal{N}$-strategies α such that $\alpha \subset \beta$. We will refer to this enumeration process as an "*outcome-shifting enumeration process*" for simplicity.

6.3 Construction

First, we define the priority tree T and assign requirements to the nodes on T as follows. Suppose $\sigma \in T$. If $|\sigma| = 3e$, then σ is assigned to the $\mathcal{N}_{i,j}$-strategy such that $e = \langle i, j\rangle$. It has three possible outcomes: ∞, f, and d, with $\infty <_L f <_L d$. If $|\sigma| = 3e+1$, then σ is assigned to the $\mathcal{S}^A_e$-strategy. If $|\sigma| = 3e+2$, then σ is assigned to the $\mathcal{S}^B_e$-strategy. In the latter two cases, σ has two possible outcomes: ∞ and f, with $\infty <_L f$.

$\mathcal{P}$ is a global requirement, and we do not put it on the tree.

We assume that K is enumerated at odd stages. That is, we fix an enumeration $\{k_{2s+1}\}_{s\in\omega}$ of K such that at each odd stage $2s+1$, exactly one number, k_{2s+1}, is enumerated into K.

In the construction, we say that an $\mathcal{N}_{i,j}$-strategy α *sees a disagreement at k at a stage s* if $k \leq s$, $[i]^{A_s}$ and $[j]^{B_s}$ agree on all arguments $\leq k$, and one of the following cases applies:

(i) s is odd (k_s enters K and we need to put $\langle k_s, 0\rangle$ into $A\cup B$). In this case, either

(1) $[i]^{A_s}(k) \neq [i]^{A_s\cup\{\langle k_s,0\rangle\}}(k)$,

(2) $[j]^{B_s}(k) \neq [j]^{B_s\cup\{\langle k_s,0\rangle\}}(k)$, or

(3) a disagreement is produced by the enumeration of the γ- or δ-uses into A or B by the initialization. For instance, there may be an $\mathcal{N}$-strategy $\alpha' \supset \alpha$ that attempts to preserve a disagreement, and the enumeration of $\langle k_s, 0\rangle$ into A or B or both (depending on α') and a one-by-one enumeration of elements of $F_A \cup F_B$ into A and B (in increasing order, as described in the $\mathcal{S}$-strategies) would lead to either $[i]^A(k) \neq [i]^{A_s}(k)$ or $[j]^B(k) \neq [j]^{B_s}(k)$. Here, F_A and F_B are the finite collections of γ-uses and δ-uses defined below outcome $\alpha'^\frown d$ after the last stage α' that produces or preserves its disagreement.

If (1) is true, then we enumerate $\langle k_s, 0\rangle$ into A. If (1) is not true but (2) is, then we enumerate $\langle k_s, 0\rangle$ into B. Otherwise, (3) is true, and we enumerate $\langle k_s, 0\rangle$ into A or B or both, according to α'. We also enumerate the corresponding numbers in $F_A \cup F_B$ into A and B respectively.

As a consequence, a disagreement between $[i]^A(k)$ and $[j]^B(k)$ is produced, and α will preserve this disagreement forever unless it is initialized later.

(ii) s is even (s is a β-expansionary stage for some $\mathcal{S}$-strategy β).

Let β be such a strategy, and let s' be the last β-expansionary stage. At stage s, to change its outcome from f to ∞, we need to enumerate all of the elements in F_A and F_B into A and B respectively. Here, F_A and F_B are the finite collections of γ-uses and δ-uses defined below outcome $\beta^\frown f$ after stage s'. Again, we enumerate these numbers into A and B in increasing order until we find that either $[i]^A(k) \neq [i]^{A_s}(k)$ or $[j]^B(k) \neq [j]^{B_s}(k)$ is true; that is, until a disagreement between $[i]^A(k)$ and $[j]^B(k)$ is produced. From now on, α will preserve this disagreement forever unless it is initialized later.

We recall that an $\mathcal{N}_{i,j}$-strategy α *preserves a disagreement at k at an odd stage s* if this disagreement was produced before and has been preserved so far (so $[i]^{A_s}(k) \neq [j]^{B_s}(k)$) and $\langle k_s, 0\rangle$ is less than one of the lengths of the truth-tables $[i](k)$ and $[j](k)$. Enumerating $\langle k_s, 0\rangle$ into $A \cup B$ causes one of the following to happen:

1. If $[i]^{A_s}(k) = [i]^{A_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into A but not into B. Both values are preserved, and the disagreement is preserved as well.

2. If $[j]^{B_s}(k) = [j]^{B_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into B but not into A. As in Case 1, the disagreement is preserved.

3. If $[i]^{A_s}(k) \neq [i]^{A_s \cup \{\langle k_s,0\rangle\}}(k)$ and $[i]^{B_s}(k) \neq [i]^{B_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into both A and B. In this case, the disagreement is again preserved, as both values are changed.

Note that whenever α produces or preserves a disagreement in this manner, all the strategies below the outcome $\alpha^\frown d$ are initialized. Such initializations can happen at most finitely often.

Formal Description of the Construction.

Stage 0: Initialize all the nodes on T and set $A_0 = B_0 = \emptyset$. Let $\Gamma^A(e, x)[0]$ and $\Delta^B(e, x)[0]$ be undefined for each e and x.

Stage $s > 0$:

Case 1: *s is odd.* We will put $\langle k_s, 0\rangle$ into $A \cup B$ at this stage.

First check whether there is an $\mathcal{N}$-strategy that sees a disagreement or needs to preserve a disagreement. Let α be the highest priority such $\mathcal{N}$-strategy. Enumerate $\langle k_s, 0\rangle$ into A or B or both accordingly. Initialize all the strategies with lower priority and do the corresponding enumerations as in (i). Otherwise, we just enumerate $\langle k_s, 0\rangle$ into A and go to the next stage.

Case 2: *s is even.* We define the approximation to the true path σ_s of length $\leq s$. Suppose that $\sigma_s \upharpoonright u$ has been defined for $u < t$ and let ξ be $\sigma_s \upharpoonright t$. We will define $\sigma_s(t)$. We have the following two subcases.

Subcase 1 ξ is an $\mathcal{N}_{i,j}$-strategy for some i and j. If ξ has produced a disagreement before and ξ has not been initialized since then, we let $\sigma_s(t) = d$. Otherwise, we check whether s is a ξ-expansionary stage. If not, then let $\sigma_s(t) = f$. If it is, then we start the outcome-shifting enumeration process to enumerate those γ-uses from F_A and δ-uses from F_B defined below the outcome $\xi^\frown f$ from the last ξ-expansionary stage into A and B respectively, one by one and in increasing order. At the same time, each time we enumerate such a number, we check whether there is an $\mathcal{N}$-strategy $\alpha \subset \xi$ that can produce a disagreement. If there is, then we stop the enumeration of F_A and F_B into A and B and let $\sigma_s = \alpha$. Declare that α produces a disagreement at stage s, let $\sigma_s = \alpha$, and go to the 'defining' phase. If not, then after all numbers in $F_A \cup F_B$ have been enumerated, we let $\sigma_s(t) = \infty$ and go to the next substage.

Subcase 2 ξ is an $\mathcal{S}_e^A$-strategy or an $\mathcal{S}_e^B$-strategy for some e. If s is not a ξ-expansionary stage, let $\sigma_s(t) = f$ and go to the next substage. Otherwise, we start the outcome-shifting enumeration process as described in Subcase 1.

Defining Phase of stage s: For those $\mathcal{S}_e^A$-strategies β with $\beta^\frown\infty \subseteq \sigma_s$, find the least y such that $\Gamma^A(e, y)$ is currently not defined, define it as 1 and let the use $\gamma(e, y)$ be a fresh number, and for those $\mathcal{S}_e^A$-strategies β with $\beta^\frown f$, find the least y such that $\Gamma^A(e, y)$ is currently not defined, define it as 0, and let the use $\gamma(e, y)$ be a fresh number. For those $\mathcal{S}_e^B$-strategies β, we define $\Delta^B(e,y)$ in the same way. Initialize all the strategies with lower priority than σ_s and go to the next stage.

Note that the enumeration of those γ-uses and δ-uses at substages into A and B ensures that those $\Gamma^A(e,x)$ and $\Delta^B(e,y)$ defined by those strategies with priority lower than σ_s are undefined.

This completes the construction.

6.4 Verification

Let $TP = \liminf_s \sigma_{2s}$ be the true path of the construction. We first prove that TP is infinite and then verify that the construction given above satisfies all the requirements. First, by the actions at the odd stages, we have that

$$k \in K \Longleftrightarrow \langle k, 0\rangle \in A \cup B,$$

and hence

Lemma 6.3. *$K \leq_{\text{tt}} A \oplus B$.*

The following lemma says that TP is infinite.

Lemma 6.4. *Let ς be any node on TP. Then*

1. *ς can only be initialized finitely often.*
2. *ς can initialize strategies with lower priority at most finitely often.*
3. *ς has an outcome $\mathcal{O}$ such that $\varsigma^\frown\mathcal{O}$ is on TP.*

Proof. We prove this lemma by induction. Let ς^- be the immediate predecessor of ς. By the induction hypothesis, there is a least stage s_0 after which ς^- can never be initialized again. Also assume that $\varsigma = \varsigma^{-\frown}\mathcal{O}'$. There are two cases.

Case 1: $\varsigma^- = \alpha$ is an $\mathcal{N}_{i,j}$-strategy for some $i,j \in \omega$.

If $\mathcal{O}'$ is d, then after stage s_0, α produces a disagreement, and this disagreement can never be destroyed as all the strategies with lower priority are initialized when this disagreement is produced. Therefore, after this, α initializes ς only when it preserves this disagreement, which can happen at most finitely often. This means that after a stage large enough that A and B have been fixed on the numbers involved in the truth-tables involved, whenever ς^- is visited at a stage s, ς is also visited at this stage, and (1) is true for ς.

If $\mathcal{O}'$ is f, then after stage s_0, there are at most finitely many α-expansionary stages. Let $s \geq s_0$ be the last α-expansionary stage. Then, after stage s, whenever ς^- is visited at a stage s, ς is also visited at this stage, and hence $\alpha^\frown\infty$ will not be visited again and ς cannot be initialized by $\alpha^\frown\infty$. If this happens, (1) is true. Note that in this case, after stage s_0, α cannot produce any disagreements as otherwise it would be preserved forever and α would have outcome d.

If $\mathcal{O}'$ is ∞, then by the choice of stage s_0, α cannot produce any disagreement after stage s_0, and ς cannot be initialized. Again, (1) is true.

(2) is obviously true since ς is an $\mathcal{S}$-strategy, which do not initialize lower priority strategies at all.

Since ς has only three outcomes, let $\mathcal{O}$ be the leftmost one that is visited infinitely often. By the construction, we never terminate the definition of σ_s at ς itself (since ς is an $\mathcal{S}$-node). Hence at almost every ς-stage (even stage), we will be able to continue the definition of σ_s beyond ς. Therefore, (3) is also true for ς.

Case 2: $\varsigma^- = \beta$ is an $\mathcal{S}_e^A$-strategy for some $e \in \omega$.

If $\mathcal{O}'$ is f, then after stage s_0, there are at most finitely many α-expansionary stages. Let $s \geq s_0$ be the last α-expansionary stage. Then after stage s, whenever β is visited, ς is also visited at this stage, and hence $\beta^\frown\infty$ cannot be visited again and ς cannot be initialized by $\beta^\frown\infty$. If $\mathcal{O}'$ is ∞, then by the choice of stage s_0, ς cannot be initialized after stage s_0. Therefore, (1) is true in both cases.

(2) is true as ς is also an $\mathcal{S}$-strategy.

(3) is true for ς for the same reason as in Case 1.

Case 3: $\varsigma^- = \beta$ is an $\mathcal{S}_e^B$-strategy for some e.

(1) is true by the same argument given in Case 2. To see (2), observe that after stage s_0, if ς does not produce any disagreement, then it does not initialize strategies with lower priority at all. Otherwise, if ς produces a disagreement at a stage $s \geq s_0$, then after stage s_0, ς can initialize at most finitely often to preserve this disagreement, and it will not initialize after some sufficiently large stage.

If (3) holds, then ς is an $\mathcal{N}_{i,j}$-strategy. By the construction, after stage s_0, any disagreement produced is preserved forever. Producing disagreement can happen at most one time, so the construction σ_s stops at ς by producing disagreement at most once. After a disagreement is produced, only preserving the disagreement can stop the construction σ_s at ς. However, there are at most finitely times that the existing disagreement must be preserved, so we will terminate the construction σ_s at ς only finitely often. Therefore, (3) is also true. □

From Lemma 6.4, we can see that any $\mathcal{N}$-strategy on TP is satisfied, and hence

Lemma 6.5. *For all $i, j \in \omega$, the requirement $\mathcal{N}_{i,j}$ is satisfied.*

Proof. Fix i and j, and let σ be the $\mathcal{N}_{i,j}$-strategy on TP. Also suppose that $[i]^A = [j]^B$ is total. We prove that $[i]^A$ is computable. Let s_0 be the last stage at which σ is initialized.

First, note that after stage s_0, σ does not produce any disagreement at all, as otherwise, as described in Lemma 6.4, the disagreement will be preserved forever, and hence $[i]^A \neq [j]^B$.

Fix n. We compute $[i]^A(n)$ by looking at the first σ-expansionary stage $s > s_0$ with $l(\sigma, s) > n$. We claim that $[i]^A(n) = [i]^{A_s}(n)$ and that $[j]^B(n) = [j]^{B_s}(n)$. Suppose not, and assume that $[i]^{A_s}(n) \neq [i]^{A_{s'}}(n)$ for some least $s' > s$ (without loss of generality, we assume that the A-side changes first). Then at stage s', we will see such a possible disagreement, and σ will produce and preserve this disagreement forever. This will contradict our assumption. □

The next lemma shows that all $\mathcal{S}$-strategies are satisfied.

Lemma 6.6. *For any $e \in \omega$, $\mathcal{S}_e^A$ and $\mathcal{S}_e^B$ are satisfied.*

Proof. Fix e. We prove that $\mathcal{S}_e^A$ and $\mathcal{S}_e^B$ are satisfied in exactly the same way. First, we show that $\mathrm{TOT}(e) = \lim_{x\to\infty} \Gamma^A(e,x)$.

Let β be the $\mathcal{S}_e^A$-strategy on TP, and let s_0 be the last stage at which β is initialized. By our construction, the $\Gamma^A(e,x)$ which are defined by β after stage s_0 are never undefined by another strategy. In fact, β defines $\Gamma^A(e,x)$ for almost all x. That is, after stage s_0, whenever β is visited, those $\Gamma^A(e,x)$ defined by strategies on the right of β are undefined, as these γ-uses have been enumerated into A. This ensures that Γ^A is total. If there are only finitely many β-expansionary stages, then after a sufficiently large stage, β defines $\Gamma^A(e,x)$ only as 0, which ensures that $\lim_x \Gamma^A(e,x) = 0$ and hence is equal to $\mathrm{TOT}(e)$. If there are infinitely many β-expansionary stages, then after stage s_0, at each β-expansionary stage, β succeeds in enumerating all those $\gamma(e,-)$-uses defined under the outcome f into A and redefines $\Gamma^A(e,x)$ to be 1, which ensures that $\lim_x \Gamma^A(e,x) = 1$ and hence is equal to $\mathrm{TOT}(e)$.

Now we show that $|\{x : \Gamma^A(e,x) \neq \Gamma^A(e,x+1)\}| \leq 2^{3^{e+1}}$. Note that in the construction, $\Gamma^A(e,x)$ may not be equal to $\Gamma^A(e,x+1)$ for some x, since a ς-strategy defining $\Gamma^A(e,x)$ is initialized when an $\mathcal{N}_{i,j}$-strategy with higher priority produces a disagreement or $\Gamma^A(e,x)$ is defined by an $\mathcal{S}_e^A$-strategy on the left of β. There are at most 3^e many such $\mathcal{N}$-strategies, and once a disagreement is produced by such an $\mathcal{N}$-strategy, it is preserved unless it is initialized later. By a simple counting argument, we know that there are at most 2^{3^e} such initializations, and hence there are no more than $2^{3^{e+1}}$ many x such that $\Gamma^A(e,x) \neq \Gamma^A(e,x+1)$. $\square$

By Lemma 6.6, we have

Lemma 6.7. TOT $\leq_{tt}$ *A', B' and hence A and B are superhigh.*

This completes the proof of Theorem 6. $\diamond\diamond$

Bibliography

[1] K. Ambos-Spies, C. G. Jockusch, Jr., R. A. Shore and R. I. Soare, *An algebraic decomposition of the recursively enumerable degrees and the coincidence of several degree classes with the promptly simple degrees*, Trans. Amer. Math. Soc. **281** (1984), 109-128.

[2] K. Ambos-Spies, A. H. Lachlan and R. I. Soare, *The continuity of cupping to* $\mathbf{0}'$, Ann. Pure Appl. Logic **64** (1993), 195-209.

[3] M. M. Arslanov, *Structural properties of the degrees below* $\mathbf{0}'$, Dokl. Akad, Nauk SSSR(N. S.) **283** (1985), 270-273.

[4] M. M. Arslanov, S. B. Cooper, A. Li, *There is no low maximal d.c.e. degree*, Math. Logic Quart. **46** (2000), 409–416.

[5] M. M. Arslanov, S. B. Cooper, A. Li, *There is no low maximal d.c.e. degree — corrigendum*, Math. Logic Quart. **50** (2004), 628–636.

[6] M. Arslanov, G. LaForte and T. Slaman, *Relative enumerability in the difference hierarchy*, J. Symbolic Logic **63** (1998), 411-420.

[7] M. Bickford and C. F. Mills, *Lowness properties of r.e. degrees*, Unpublished preprint, 1982.

[8] S. Binns, B. Kjos-Hanssen, M. Lerman and R. Solomon, *On a conjecture of Dobrinen and Simpson concerning almost everywhere domination*, J. Symbolic Logic, **71** (2006), 119-136.

[9] D. Cenzer, G. LaForte and G. Wu, *The nonisolating degrees are nowhere dense in the computably enumerable degrees*. To appear.

[10] D. Cenzer, N. Y. Franklin, J. Liu and G. Wu, *Embedding the diamond lattice in c.e. tt-degrees with superhigh atoms*, Lecture Notes in Computer Science **5532** (2009), 420-429.

[11] P. Cholak, M. Groszek and T. A. Slaman, *An almost deep degree*, J. Symbolic Logic, **66** (2001), 881-901.

[12] S. B. Cooper, *Degrees of unsolvability*, Ph.D. thesis, Leicester University, Leicester, England.

[13] S. B. Cooper, *Degrees of unsolvability complementary between recursively enumerable degrees (I)*, Ann. Math. Logic, **4** (1972), 31-73.

[14] S. B. Cooper, *Minimal pairs and high recursively enumerable degrees*, J. Symbolic Logic, **39** (1974), 655-660.

[15] S. B. Cooper, *On a theorem of C. E. M. Yates*, Handwritten notes, 1974.

[16] S. B. Cooper, L. A. Harrington, A. H. Lachlan, S. Lempp and R. I. Soare, *The d.r.e. degrees are not dense*, Ann. Pure Appl. Logic, **55** (1991), 125-151.

[17] S. B. Cooper, S. Lempp, and P. Watson, *Weak density and cupping in the d.r.e. degrees*, Israel J. Math., **67** (1989), 137–152.

[18] S. B. Cooper and A. Li, *Splitting and Nonsplitting*, II: A Low_2 c.e. degree above which $\mathbf{0}'$ is not splittable, J. Symbolic Logic, **67** (2002), 1391-1430.

[19] S. B. Cooper and A. Li, *On Lachlan's major subdegree problem*, Arch. Math. Logic **47** (2008), 341-434.

[20] S. B. Cooper and X. Yi, *Isolated d.c.e. degrees*, University of Leeds, Dept. of Pure Math., Preprint series, No. 17, 1995.

[21] M. Davis, *Computability and Unsolvability*, New York, McGraw-Hill, 1958.

[22] A. N. Degtev, *tt- and m-degrees*, Algebra i Logika, **12** (1973), 143-161. English Translation in Algebra and Logic **12** (1973), 78-89.

[23] D. Ding and L. Qian, *Isolated d.r.e. degrees are dense in r.e. degree structure*, Arch. Math. Logic **36** (1996), 1-10.

[24] N. L. Dobrinen and S. G. Simpson, *Almost everywhere domination*, J. Symbolic Logic, **69** (2004), 914-922.

[25] R. Downey, *D.r.e. degrees and the nondiamond theorem*, Bull. London Math. Soc. **21** (1989), 43-50.

[26] R. L. Epstein, *Minimal degrees of unsolvability and the full approximation construction*, Mem. Amer. Math. Soc., **162**, 1975.

[27] R. L. Epstein, *Initial segments of degrees below* $\mathbf{0}'$, Mem. Amer. Math. Soc., **30**, 1981.

[28] R. L. Epstein, R. Hass and R. L. Kramer, *Hierarchies of sets and degrees below* $\mathbf{0}'$, Lect. Notes Math., **859** (1981), 32-48.

[29] Yu. L. Ershov, *A hierarchy of sets, I*, Algebra i Logika, **7** (1968), 1-27. English Translation, Consultants Bureau, YN, 25-43.

[30] Yu. L. Ershov, *A hierarchy of sets, II*, Algebra i Logika, **7** (1968), 15-47. English Translation, Consultants Bureau, YN, 212-232.

[31] Yu. L. Ershov, *A hierarchy of sets, III*, Algebra i Logika, **7** (1970), 34-51. English Translation, Consultants Bureau, YN, 20-31.

[32] Yu. L. Ershov, S. S. Goncharov, A. Nerode and J. B. Remmel (eds.), *Handbook of Recursive Mathematics*, volume 138 of Stud. Logic Found. Math., North-Holland, Amsterdam, 1998.

[33] P. A. Fejer and R. A. Shore, *A direct construction of a minimal recursively enumerable truth-table degree*, in: Recursion Theory Week, Proceedings Oberwolfach (1989) K. Ambos-Spies, G. H. Muller and G. E. Sacks, eds. Springer-Verlag, LNMS 1432 (1990), 187-204.

[34] P. A. Fejer and R. I. Soare, *The plus-cupping theorem for the recursively enumerable degrees*, In Logic Year 1979-80: University of Connecticut (1981), 49-62.

[35] R. M. Friedberg, *Two recursively enumerable sets of incomparable degrees of unsolvability*, Proc. Nat. Ac. Sci. **43** (1957), 236-238.

[36] K. Gödel, *Über formal unentscheidbare sätze der Principia Mathematica und verwandter systeme. I*, Monatsh. Math. Phys., **38** (1931), 173-178.

[37] L. A. Harrington, *Plus-cupping in the recursively enumerable degrees*, Notes, 1978.

[38] L. A. Harrington, *Understanding Lachlan's monster paper*, Notes, 1980.

[39] L. A. Harrington and S. Shelah, *The undecidability of the recursively enumerable degrees*, Bull. Amer. Math. Soc. (N.S.) **6**, (1982), 79-80.

[40] S. Ishmukhametov, *D.r.e. sets, their degrees and index sets*, Ph.D. thesis, Novosibirsk, Russia, 1986.

[41] S. Ishmukhametov and G. Wu, *Isolation and the high/low hierarchy*, Arch. Math. Logic, accepted.

[42] C. G. Jockusch, Jr., A. Li, and Y. Yang, *A join theorem for the computably enumerable degrees*, Trans. Amer. Math. Soc. **356** (2004), 2557-2568.

[43] C. G. Jockusch, Jr. and J. Mohrherr, *Embedding the diamond lattice in the recursively enumerable truth-table degrees*, Proc. Amer. Math. Soc. **94** (1985), 123-128.

[44] D. Kaddah, *Infima in the d.r.e. degrees*, Ann. Pure Appl. Logic, **62** (1993), 207–263.

[45] A. H. Lachlan, *On a problem of G. E. Sacks*, Proc. Amer. Math. Sco. **16** (1965), 972-979.

[46] A. H. Lachlan, *Lower bounds for pairs of recursively enumerable degrees*, Proc. London Math. Soc. **16** (1966), 537-569.

[47] A. H. Lachlan, *On the lattice of recursively enumerable sets*, Trans. Amer. Math. Soc. **130** (1968), 1-37.

[48] A. H. Lachlan, *A recursively enumerable degree which will not split over all lesser ones*, Ann. Math. Logic **9** (1975), 307-305.

[49] G. LaForte, *The isolated d.r.e. degrees are dense in the r.e. degrees*, Math. Logic Quart. **42** (1996), 83-103.

[50] S. Lempp and T. A. Slaman, *A limit on relative genericity in the recursively enumerable sets*, J. Symbolic Logic **54** (1989), 376-395.

[51] A. E. M. Lewis, *A single minimal complement for the c.e. degrees*, Trans. Amer. Math. Soc. **359** (2007), 5817-5865.

[52] A. Li, Y. Song and G. Wu, *Universal cupping degrees*, Lecture Notes in Computer Science, **3959** (2006), 721-730.

[53] A. Li, G. Wu and Z. Zhang, *A hierarchy for cuppable degrees*, Illinois J. Math. **44** (2000), 619-632.

[54] J. Liu, S. Wang and G. Wu, *Infima of d.r.e. degrees*, Arch. Math. Logic **49** (2009), 17-34.

[55] J. Liu and G. Wu, *Isolation, infima and diamond embeddings*, J. Logic and Comp. **17** (2007), 1153-1166.

[56] J. Liu and G. Wu, *Joining to high degrees via noncuppables*, Arch. Math. Logic **49** (2010), 195-211.

[57] J. Liu and G. Wu, *Almost universal cupping degrees*, submitted.

[58] S. S. Marchenkov, *The existence of enumerable minimal truth table degrees*, Algebra i Logika, **14** (1975), 257-261. English Translation in Algebra and Logic **14** (1975), 257-261.

[59] S. S. Marchenkov, *On the comparison of the upper semilattice of r.e. m-degrees and tt-degrees*, Mat. Zam. **20** (1976), 19-26.

[60] D. Martin, *Classes of recursively enumerable sets and degrees of unsolvability*, Z. Math. Logik Grundlag. Math. **12** (1966), 295-310.

[61] D. Miller, *High recursively enumerable degrees and the anti-cupping property*, In Logic Year 1979-80: University of Connecticut (1981), 230-245.

[62] J. Mohrherr, *Density of a final segment of the truth-table degrees*, Pac. J. Math. **115** (1984), 409-419.

[63] J. Mohrherr, *A refinement of* low_n *and* $high_n$ *for the r.e. degrees*, Z. Math. Logik Grundlag. Math. **32** (1986), 5-12.

[64] A. A. Muchnik, *Solution of Post's reduction problem and of certain other problems in the theory of algorithms*, Trudy Moskov Mat. Obsc. **7** (1957), 391-405.

[65] A. Nerode, *General topology and partial recursive functions*, Talks Cornell Summ. Inst. Symb. Logic, Cornell (1957), 247-251.

[66] K. M. Ng, *On very high degrees*, J. Symbolic Logic, **73** (2008), 309-342.

[67] P. Odifreddi, *Classical recursion theory I*, North-Holland, 1989.

[68] P. Odifreddi, *Classical recursion theory II*, North-Holland, 1999.

[69] D. Posner and W. Robinson, *Degrees joining to* $\mathbf{0}'$, J. Symbolic Logic, **46** (1981), 714-722.

[70] E. L. Post, *Recursively enumerable sets of positive integers and their decision problem*, Bull. Amer. Math. Soc. **50** (1944), 284-316.

[71] G. E. Sacks, *Degrees of unsolvability*, Ann. Math. Stud. 55, Princeton University Press, 1963.

[72] G. E. Sacks, *On the degrees less than* $\mathbf{0}'$, Ann. Math. **77** (1963), 211-231.

[73] G. E. Sacks, *The recursively enumerable degrees are dense*, Ann. Math. **80** (1964), 300-312.

[74] G. E. Sacks, *On a theorem of Lachlan and Martin*, Proc. Amer. Math. Soc. **18** (1967), 140-141.

[75] D. Seetapun, *Contributions to recursion theory*, Ph.D. thesis, Trinity College, Cambridge, 1991.

[76] D. Seetapun and T. A. Slaman, *Minimal complements*, manuscript.

[77] J. R. Shoenfield, *Undecidable and creative theories*, Fund. Math. **49** (1961), 171-179.

[78] J. R. Shoenfield, *Application of model theory to degrees of unsovability*, in: Theory of Models (Symposium on the theory of models, J. Addison, L. Henkin and A. Tarski eds.), North-Holland, Amsterdam (1965), 359-363.

[79] R. A. Shore and T. A. Slaman, *Working below low*$_2$ *recursively enumerable degrees*, Arch. Math. Logic, **29** (1990), 201-211.

[80] R. A. Shore and T. A. Slaman, *Working below a high recursively enumerable degree*, J. Symbolic Logic, **58** (1993), 824-859.

[81] S. G. Simpson, *Almost everywhere domination and superhighness*, Math. Logic Quart. **53** (2007), 462-482.

[82] T. A. Slaman and J. Steel, *Complementation in the Turing degrees*, J. Symbolic Logic **54** (1989), 160-176.

[83] R. I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Heidelberg, 1987.

[84] R. I. Soare, *The history and concept of computability*, in: *Handbook of Computability Theory*, volume 140 of Stud. Logic Found. Math., North-Holland, Amsterdam, (1999), 3-36.

[85] B. A. Trakhtenbrot, *Tabular representation of recursive operators*, Dokl. Acad. Nauk **101** (1955), 417-420.

[86] A. M. Turing, *On computable numbers, with an application to Entscheidungsproblem*, Proc. London Math. Soc. Ser. 2, 42 (Parts 3 and 4) (1936), 230-265.

[87] G. Wu, *Isolation and lattice embeddings*, J. Symbolic Logic **67** (2002), 409-420.

[88] G. Wu, *Bi-isolation in the d.r.e. degrees*, J. Symbolic Logic **69** (2004), 1055-1064.

[89] G. Wu, *Quasi-complements of the cappable degrees*, Math. Logic Quart. **50** (2004), 189-201.

[90] G. Wu, *Jump operator and Yates degrees*, J. Symbolic Logic **71** (2006), 252-264.

[91] C. E. M. Yates, *A minimal pair of recursively enumerable degrees*, J. Symbolic Logic **31** (1966), 159-168.

Publications

[1] D. Cenzer, N. Y. Franklin, J. Liu and G. Wu, *Embedding the diamond lattice in c.e. tt-degrees with superhigh atoms*, Lecture Notes in Computer Science **5532** (2009), 420-429.

[2] J. Liu, S. Wang and G. Wu, *Infima of d.r.e. degrees*, Arch. Math. Logic **49** (2009), 17-34.

[3] J. Liu and G. Wu, *Isolation, infima and diamond embeddings*, J. Logic and Comp. **17** (2007), 1153-1166.

[4] J. Liu and G. Wu, *Joining to high degrees via noncuppables*, Arch. Math. Logic **49** (2010), 195-211.

[5] J. Liu and G. Wu, *Almost universal cupping degrees*, submitted to J. Symbolic Logic.